Invariant Measure of Quantum Trajectories

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work in collaboration with
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- **1) Generalities**
- **2) Invariant measure**
- **3) Speed of convergence**

1) Generalities

1. Definition-First properties
2. Link with other works
3. Examples

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Framework

- Let $\mathcal{H}_S = \mathbb{C}^k$ be the space describing the quantum system $\mathcal{S}$
- Let $\mathcal{D}(\mathcal{S})$ be the set of **density matrices** on $\mathcal{S}$, that is

$$\mathcal{D}(\mathcal{S}) = \{\rho \in \mathcal{B}(\mathbb{C}^k),\ s.t,\ \rho \geqslant 0, \mathrm{tr}(\rho) = 1\}$$

$\Rightarrow$ The **rank one projectors** belonging to $\mathcal{D}(\mathcal{S})$ represent **the pure states.**

- We denote by (ρ_n) the Markov chain describing the evolution of $\mathcal{S}$ undergoing repeated indirect measurements. If $\rho_n = \rho$ then ρ_{n+1} can takes ℓ different values

$$\rho_{n+1}(i) = \frac{V_i \rho V_i^*}{\mathrm{tr}(V_i \rho V_i^*)}$$

with probability $\mathrm{tr}(V_i \rho V_i^*)$.

- The sequence (ρ_n) is called a **quantum trajectory.**

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First properties

- **Expectation:** Let $\mathcal{S}_n$ be the natural filtration of (ρ_n), that is $\mathcal{S}_n = \sigma(\rho_1, \dots, \rho_n)$, we have

$$\mathbb{E}[\rho_{n+1}|\mathcal{S}_n] = \phi(\rho_n),$$

where

$$\phi(\rho) = \sum_{i=1}^{\ell} V_i \rho V_i^*.$$

The map ϕ is a **completely positive and trace preserving map (CPTP)** which describes the evolution of $\mathcal{S}$ without measurement. In particular

$$\mathbb{E}[\rho_n] = \phi^n(\rho_0).$$

- **Pure states:** If $\rho_0 = |x_0\rangle\langle x_0|$ is a pure state, then there exists (x_n) such that $\rho_n = |x_n\rangle\langle x_n|$. If $x_n = x$ then x_{n+1} can takes ℓ different values

$$x_{n+1} = \frac{V_i x}{\|V_i x\|}, i = 1, \dots, \ell$$

with probability $\|V_i x\|^2 = \mathrm{tr}(V_i |x\rangle\langle x| V_i^*)$

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Purification

- **Pur:** The only projectors Q statisfying $QV_i^*V_iQ = \lambda_i Q, i = 1, \ldots, \ell$ are rank one projectors.
- Under this condition, we have the following theorem due to H. Maassen and B. Kümmerer

Theorem

Assume (Pur). The Markov chain (ρ_n) purifies in the sense that for all $k \in \mathbb{N}^*$

$$\lim_{n \mapsto \infty} tr(\rho_n^k) = 1$$

- Note that $\mathrm{tr}(\rho^2) = 1$ implies that ρ is a pure state
- Under (Pur) one can show that there exists (y_n) such that

$$\lim_{n \mapsto \infty} \left\| \rho_n - |y_n\rangle\langle y_n| \right\| = 0$$

Existence of an invariant measure

- In the sequel we shall always assume (Pur). Then one can restrict the study of the problem of invariant measure to the set of pure states.
- The set of pure states is isomorphic to $P(\mathbb{C}^k)$ the projective space over $\mathbb{C}^k$. For any non zero $x \in \mathbb{C}^k$, we denote $\hat{x}$ it's representent on $P(\mathbb{C}^k)$
- In order to simplify the notation we put $A \cdot \hat{x} = \widehat{A\hat{x}}$ when $\|A\hat{x}\| \neq 0$

Definition

The Markov kernel describing the quantum trajectory attached to $\{V_i, i = 1, \ldots, \ell\}$ is:

$$\Pi(\hat{x}, A) = \sum_{i=1}^{\ell} \mathbf{1}_A(V_i \cdot \hat{x}) \|V_i \hat{x}\|^2,$$

for all Borel set A of $P(\mathbb{C}^k)$.

Existence of an invariant measure

- Note that the Markov kernel π is Feller, that is it maps continuous functions into continuous functions.

$$\begin{aligned}
\Pi f(\hat{x}) &= \int_{P(\mathbb{C}^k)} f(\hat{y})\Pi(\hat{x}, d\hat{y}) \\
&= \sum_{i=1}^{\ell} f(V_i \cdot \hat{x})\|V_i\hat{x}\|^2 = \sum_{i=1}^{\ell} f\left(\frac{V_i\hat{x}}{\|V_i\hat{x}\|}\right)\|V_i\hat{x}\|^2
\end{aligned}$$

- Note that this Markov kernel is not strongly Feller.
- Denote $\mathcal{I}nv$ the set of invariant measure of Π ($\nu\Pi = \nu$)

Proposition

Since the Markov kernel is Feller and the set $P(\mathbb{C}^k)$ is compact, we can apply the Markov Kakutani Theorem to conclude that $\mathcal{I}nv$ is non-empty.

- Question: Do we have uniqueness of invariant measure and do we have convergence towards this invariant measure?

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Products of random matrices

- Recall that the quantum trajectory can be defined as

$$\hat{x}_n = V_{i_n} \dots V_{i_1} \cdot \hat{x}_0$$

  In some sense we can summarize the quantum trajectory through the product of random matrices $V_{i_n} \dots V_{i_1}$

- This is exactly the setup of product of random matrices developed by *Y. Guivarc'h, A. Raugi, E. Lepage, P. Bougerol, J. Lacroix...*. **"Products of Random Matrices with Applications to Schrödinger Operators"** (book by P. Bougerol and J. Lacroix)

- In this setup the main assumptions are

  1. The matrices V_i are invertible
  2. The matrices V_i are *i.i.d* (extended to Markovian dependance)
  3. The matrices V_i are strongly irreducible, there is no $\cup_{i=1}^{p} E_i$ such that

$$V_j \cup_{i=1}^{p} E_i = \cup_{i=1}^{p} E_i, \; j = 1, \dots, \ell$$

- We won't use such assumptions.

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Example 1

- Reproducing classical Markov chains. Let (X_n) be a Markov chain valued on $E = \{1, \dots, N\}$ with transition matrix $P = (p_{ij})$. Then defining

$$V_{ij} = \sqrt{p_{ij}} E_{ij} = \sqrt{p_{ij}} |e_i\rangle\langle e_j|.$$

after one step one can see that the quantum trajectory $\hat{x}_n$ is valued in $\{e_i, i = 1, \dots, k\}$.

- If the Markov chain (X_n) is irreducible and aperiodic then the quantum trajectory $(\hat{x}_n)$ converges in law to the invariant measure

$$\sum_{i=1}^{\ell} \pi(i) \delta_{e_i},$$

where $\pi(.) = (\pi(i))_{i=1,\dots,l}$ denote the unique invariant measure of (X_n).

- Note that the matrices V_{ij} are non invertible

Example 2

- Let $V_1 = \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$ and $V_2 = \begin{pmatrix} 0 & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{2}}{\sqrt{3}} & 0 \end{pmatrix}$

$$e_1 \mapsto \begin{cases} e_1 \text{ with probability } \frac{1}{3} \\ e_2 \text{ with probability } \frac{2}{3} \end{cases}$$

$$e_2 \mapsto \begin{cases} e_1 \text{ with probability } \frac{1}{4} \\ e_2 \text{ with probability } \frac{3}{4} \end{cases}$$

- Again it reproduces a classical Markov chain.
- if you start with $\hat{x}_0 = (x, y)$ with $x^2 + y^2 = 1$

$$(x, y) \mapsto \begin{cases} \left(\frac{\frac{1}{\sqrt{3}}x}{\sqrt{\frac{1}{3}x^2 + \frac{1}{4}y^2}}, \frac{\frac{1}{2}y}{\sqrt{\frac{1}{3}x^2 + \frac{1}{4}y^2}} \right) & \text{with probability } \frac{1}{3}x^2 + \frac{1}{4}y^2 \\ \left(\frac{\frac{\sqrt{3}}{2}y}{\sqrt{\frac{2}{3}x^2 + \frac{3}{4}y^2}}, \frac{\frac{\sqrt{2}}{\sqrt{3}}x}{\sqrt{\frac{2}{3}x^2 + \frac{3}{4}y^2}} \right) & \text{with probability } \frac{2}{3}x^2 + \frac{3}{4}y^2 \end{cases}$$

- Note that the matrices V_i are non strongly irreducible.

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- Again it reproduces a classical Markov chain.
- The uniform measure is invariant

$$\frac{1}{2}\delta_{e_1} + \frac{1}{2}\delta_{e_2}$$

- But also

$$\delta_{\frac{1}{\sqrt{2}}(e_1+e_2)}$$

is also invariant

- Note that purification is not satisfied since $V_i^* V_i = \frac{1}{2} Id$.

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Example 3

- Let $V_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $V_2 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$
- It does not reproduce any classical Markov chain on a finite set
- Note that since V_1 and V_2 are real it preserves real vectors
- Simulations show that the invariant measure has full support on the set of real vectors
- Remark: The support of invariant measure is a tedious question. In the classical setup of product of i.i.d random matrices, knowing if it is absolutely continuous with respect to the Lebesgue measure is still an open question.

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Assumptions

- Our aim: To show that there exist a unique invariant measure
- Two conditions
  1. **Pur:** The only projectors Q statisfying $QV_i^* V_i Q = \lambda_i Q, i = 1, \dots, l$ are rank one projectors.
  2. **Irr:** There exists a unique minimal projector Q such that

$$V_i Q(\mathbb{C}^k) \subset Q(\mathbb{C}^k), i = 1, \dots, \ell$$

- The condition (Irr) is equivalent to the fact that the quantum channel ϕ has a unique invariant state. We denote ρ_{inv} this state

$$\phi(\rho_{inv}) = \rho_{inv}$$

Theorem

Assume (Pur) and (Irr), the Markov kernel associated to $\{V_i\}$ has a unique invariant measure denoted ν_{inv} on $P(\mathbb{C}^k)$.

- Note that all the good previous examples satisfy (Pur) and (Irr).

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Probability space

- The probability space is composed of two parts, the evolution of the quantum trajectory $\hat{x}_n$ and the results of measurements.
- Let $\mathcal{A} = \{1, \dots, \ell\}$ and $\Omega := \mathcal{A}^{\mathbb{N}}$. Let $(\mathcal{F}_n)_n$ be the filtration generated by the cylinder sets $\Lambda_{I_n} = \Lambda_{i_1,\dots,i_n} = \{\omega \in \Omega | \omega_1 = i_1, \dots, \omega_n = i_n\}$ and let $\mathcal{F}$ be the smallest σ-algebra such that $\mathcal{F}_n \subset \mathcal{F}$ for all $n \in \mathbb{N}$.
- Let $\mathcal{G} = \mathcal{F} \otimes \mathcal{B}$, then $(\Omega \times P(\mathbb{C}^d), \mathcal{G})$ is a measurable compact space.
- Set $\mathcal{G}_{\mathcal{F}} = \mathcal{F} \times \{\emptyset, P(\mathbb{C}^d)\}$. Then any $\mathcal{G}$ measurable function such that $f(\omega, \hat{x}) = f(\omega, \hat{y})$ for any $\omega \in \Omega$ and $\hat{x}, \hat{y} \in P(\mathbb{C}^k)$ is $\mathcal{G}_{\mathcal{F}}$–measurable. There is a natural embedding of $\mathcal{F}$ measurable functions into $\mathcal{G}_{\mathcal{F}}$–measurable functions setting $f : \omega, \hat{x} \mapsto f(\omega)$.
- Let ν be a probability measure over $(P(\mathbb{C}^d), \mathcal{B})$. We extend it to a measure μ_ν over $(\Omega \times P(\mathbb{C}^d), \mathcal{G})$ setting, for any $A \in \mathcal{B}$ and any $I_n \in \mathcal{A}^n$,

$$\mu_\nu(I_n \times A) := \int_A \|V_{I_n}\hat{x}\|^2 d\nu(\hat{x}),$$

with $V_{I_n} = V_{i_n} \cdots V_{i_2} V_{i_1}$.

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- Set $\mathcal{G}_{\mathcal{F}} = \mathcal{F} \times \{\emptyset, P(\mathbb{C}^d)\}$. Then any $\mathcal{G}$ measurable function such that $f(\omega, \hat{x}) = f(\omega, \hat{y})$ for any $\omega \in \Omega$ and $\hat{x}, \hat{y} \in P(\mathbb{C}^k)$ is $\mathcal{G}_{\mathcal{F}}$–measurable. There is a natural embedding of $\mathcal{F}$ measurable functions into $\mathcal{G}_{\mathcal{F}}$–measurable functions setting $f : \omega, \hat{x} \mapsto f(\omega)$.
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$$\mu_\nu(I_n \times A) := \int_A \|V_{I_n}\hat{x}\|^2 d\nu(\hat{x}),$$

with $V_{I_n} = V_{i_n} \cdots V_{i_2} V_{i_1}$.

- Let $\hat{y} \in P(\mathbb{C}^d)$ be fixed. Let the sequence $(\hat{x}_n)_{n\in\mathbb{N}}$ be defined by

$$\hat{x}_n := \widehat{V_{I_n} x}, \text{if } \|V_{I_n} x\| \neq 0 \quad \text{and} \quad \hat{x}_n := \hat{y}, \text{else}.$$

  The second eventuality has μ_ν probability 0. The sequence $(\hat{x}_n)_n$ is a realisation on $(\Omega \times P(\mathbb{C}^d), \mathcal{G}, \mu_\nu)$ of the Markov chain defined by Π and initial probability measure ν.
- Particularly, under the law μ_ν, for each $n \in \mathbb{N}$, $\hat{x}_n$ has law $\nu\Pi^n$.

Proposition

Let ν be a measure over $(P(\mathbb{C}^d), \mathcal{B})$, we have

$$\mathbb{E}_\nu(|\hat{x}\rangle\langle\hat{x}|) \quad = \quad \rho_\nu \in \mathcal{D}(\mathcal{S}). \tag{1}$$

The marginal of μ_ν restricted to $\mathcal{G}_\mathcal{F}$ is the probability measure $\mathbb{P}_{\rho_\nu}$ over $(\Omega, \mathcal{F})$:

$$\mathbb{P}_{\rho_\nu}(I_n) := tr(V_{I_n}^* V_{I_n} \rho_\nu),$$

for any $n \in \mathbb{N}$ and any $I_n \in \mathcal{A}^n$.

Probability space

- Let $\hat{y} \in P(\mathbb{C}^d)$ be fixed. Let the sequence $(\hat{x}_n)_{n\in\mathbb{N}}$ be defined by

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- The last proposition implies the following one

Proposition

Assume (Irr). For all $\nu \in \mathcal{I}nv$ we have

$$\rho_{inv} = \mathbb{E}_{\nu}(|\hat{x}_0\rangle\langle\hat{x}_0|).$$

For all ν_a and ν_b in $\mathcal{I}nv$ and all f L^1 $\mathcal{F}$-measurable random variable

$$\mathbb{E}_{\nu_a}(f) = \mathbb{E}_{\nu_b}(f)$$

In particular the law of f does not depend on the choice $\nu \in \mathcal{I}nv$.

- The next aim is to show that $(\hat{x}_n)$ is close to an $\mathcal{F}$ measurable process when n goes to infinity.

- The key martingale is defined as follows

Proposition

Let us consider the sequence of random variables generated by the chaotic state

$$M_n := \frac{V_{I_n}^* V_{I_n}}{tr(V_{I_n}^* V_{I_n})}, \ \textit{if } tr(V_{I_n}^* V_{I_n}) \neq 0 \quad \textit{and} \quad M_n := I_k/k, \ \textit{else},$$

converges almost surely and in L^1–norm to a $\mathcal{G}_{\mathcal{F}}$ measurable random variable M_∞. In particular this martingale yields the following change of measure formula

$$\left.\frac{dP_\rho}{dP_{ch}}\right|_{\mathcal{F}_n} = k\, tr(\rho M_n), \quad \left.\frac{dP_\rho}{dP_{ch}}\right|_{\mathcal{F}} = k\, tr(\rho M_\infty) \tag{2}$$

Moreover, the condition (Pur) implies that there exists a $\mathcal{F}$ measurable $\hat{z}$ such that

$$M_\infty = |\hat{z}\rangle\langle\hat{z}|.$$

The key martingale

- Recall that $\hat{x}_n = V_{I_n} \cdot \hat{x}_0$. Let $U_n D_n$ be the polar decomposition of V_{I_n}.
- The operator D_n is proportional to $M_n^{1/2}$
- Let us note that
$$\lim_{n \mapsto \infty} M_n^{1/2} \cdot \hat{x} = \hat{z}, P_{|\hat{x}\rangle\langle\hat{x}|} a.s.$$
Since M_n converges to $P_{|\hat{z}\rangle\langle\hat{z}|}$ it remains to show that $\hat{x}$ is not orthogonal to $\hat{z}$. This comes from the fact that
$$dP_{|\hat{x}\rangle\langle\hat{x}|} = k|\langle\hat{x}, \hat{z}\rangle|^2 dP_{ch}$$

- Introducing the natural distance
$$d(x,y) = \sqrt{1 - \frac{|\langle x, y\rangle|}{\|x\|_2 \|y\|_2}}$$
we get
$$\lim_{n \mapsto \infty} d(\hat{x}_n, U_n \cdot \hat{z}) = 0, a.s$$
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Uniqueness of the invariant measure

- This gives the uniqueness of the invariant measure

Proposition

The set $\mathcal{I}nv$ contains a unique element denoted ν_{inv}

3) Speed of convergence

1. Exponential convergence of $\mathcal{F}$ measurable functions
2. Exponential convergence towards a $\mathcal{F}$ mesurable function
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Primitivity

- In the sequel we shall assume the following condition
  Prim The quantum channel is primitive, that is 1 is the only eigenvalue of modulus one (the invariant state is faithfull).
- In particular we have the following convergence

Proposition

There exists $C > 0$ *and* $0 < \lambda < 1$ *such that for all* $k \in \mathbb{N}^*$ *and all* $\rho \in \mathcal{D}(\mathcal{S})$

$$\|\phi^k(\rho) - \rho_{inv}\| \leqslant C\lambda^k.$$

- Note that without (*Prim*) there still exists m such that

$$\|\frac{1}{m}\sum_{r=0}^{m-1}\phi^{mk+r}(\rho) - \rho_{inv}\| \leqslant C\lambda^k$$

Convergence of $\mathcal{F}$ mesurable functions

- Recall the total variation distance is defined as

$$\|\mu - \nu\|_{TV} = \sup_{A \in \mathcal{F}} |\mu(A) - \nu(A)|$$

Proposition

Assume (Prim). There exists $C > 0$, $0 < \lambda < 1$ such that for all $\rho \in \mathcal{D}(\mathcal{S})$ and for all $n \in \mathbb{N}^*$

$$\sup_{A \in \mathcal{F}} \left| \mathbb{E}_{\phi^n(\rho)}[\mathbf{1}_A] - \mathbb{E}_{\rho_{inv}}[\mathbf{1}_A] \right| \leqslant C\lambda^n \quad (3)$$

$$\left\| \mathbb{P}_{\phi^n(\rho)} - \mathbb{P}_{\rho_{inv}} \right\|_{TV} \leqslant C\lambda^n \quad (4)$$

- We shall use this proposition in the following form. Let f be a $\mathcal{F}$ measurable function bounded by 1

$$|\mathbb{E}_\rho [f \circ \theta^n] - \mathbb{E}_{\rho_{inv}}[f]| \leqslant C\lambda^n \quad (5)$$

where θ denotes the usual shift on Ω.

Convergence towards a $\mathcal{F}$ mesurable process

- In order to prove the exponential convergence we need to introduce a new process which is statistically close to $\hat{x}_n$

Definition

For all $n \in \mathbb{N}^*$ and for all $I_n \in \mathcal{A}^n$ define

$$y_{I_n} = \operatorname*{argmax}_{\hat{x} \in P(\mathbb{C}^k)} \{\ln(\mathbb{P}_{\hat{x}}(I_n))\} \tag{6}$$

and put

$$\hat{z}_n = V_{I_n} \cdot \hat{y}_{I_n}$$

- Note that $\mathbb{P}_{\hat{x}}(I_n) = \|V_{I_n}\hat{x}\|^2$ then

$$\sqrt{V_{I_n}^* V_{I_n}}\hat{y}_{I_n} = a_1(V_{I_n})\hat{y}_{I_n},$$

  where $a_1(X)$ denote the largest singular value of X.

- In particular $\hat{z}_n$ is $\mathcal{F}$ mesurable and

$$\lim d(\hat{x}_n, \hat{z}_n) = 0.$$

Convergence towards a $\mathcal{F}$ mesurable process

- The following proposition expresses the exponential convergence towards a $\mathcal{F}$ measurable process.

Proposition

There exists $0 < \lambda < 1$ *and* $C > 0$ *such that for any probability measure* ν *over* $(P(\mathbb{C}^k), \mathcal{B})$,

$$\mathbb{E}_\nu(d(\hat{x}_n, \hat{z}_n)) \leqslant C\,\lambda^n.$$

More generally for any probability measure ν *over* $(P(\mathbb{C}^k), \mathcal{B})$*, for all* $k \in \mathbb{N}$

$$\mathbb{E}_\nu(d(\hat{x}_{n+k}, \hat{z}_n \circ \theta^k)) \leqslant C\,\lambda^n. \tag{7}$$

- In order to show such a result we shall use the exterior product $\wedge$ on $P(\mathbb{C}^k)$

$$d(\hat{x}, \hat{y}) = \frac{\|x \wedge y\|}{\|x\|\|y\|}. \tag{8}$$

Convergence towards a $\mathcal{F}$ mesurable process

- First we remark that

$$d(\hat{x}_n, \hat{z}_n) = d(V_{I_n} \cdot \hat{x}, V_{I_n} \cdot \hat{y_{I_n}}) = \frac{\| \wedge^2 V_{I_n} x \wedge y_{I_n} \|}{\| V_{I_n} x \| \| V_{I_n} \|} \leqslant \frac{\| \wedge^2 V_{I_n} \|}{\| V_{I_n} x \|^2}$$

- Next we show that

$$\mathbb{E}_\nu \left(\frac{\| \wedge^2 V_{I_n} \|}{\| V_{I_n} x \|^2} \right) \leqslant \sum_{I_n \in \mathcal{A}^n} \| \wedge^2 V_{I_n} \|$$

- In fact

$$f(n) = \sum_{I_n \in \mathcal{A}^n} \| \wedge^2 V_{I_n} \| = \mathbb{E}_{ch} \left[\frac{\| \wedge^2 V_{I_n} \|}{P_{ch}[I_n]} \right]$$

defines a sub-multiplicative function which tends to 0 when n goes to infinity.

- Then we get

$$\mathbb{E}_\nu (d(\hat{x}_n, \hat{z}_n)) \leqslant f(n) \leqslant C \lambda^n.$$

Wasserstein

- Recall that the Wassersein distance can be viewed as

$$W_1(\mu,\nu) = \sup_{f\in Lip(1)} \Big| \int_\Omega f\, d(\mu-\nu)\Big|,$$

- Let $f \in Lip(1, P(\mathbb{C}^k))$, let $n \in \mathbb{N}^*$. We shall distinguish the case where $n = 2k$ and $n = 2k+1$. Let us start with the case where $n = 2k$, we have

$$\begin{aligned}
& \Big|\mathbb{E}_\nu[f(\hat{x}_n)] - \mathbb{E}_{\nu_{inv}}[f(\hat{x})]\Big| \\
\leqslant \quad & \Big|\mathbb{E}_\nu[f(\hat{x}_{2k})] - \mathbb{E}_\nu[f(\hat{z}_k \circ \theta^k)]\Big| + \Big|\mathbb{E}_\nu[f(\hat{z}_k \circ \theta^k)] - \mathbb{E}_{\nu_{inv}}[f(\hat{z}_k \circ \theta^k)]\Big| \\
& + \Big|\mathbb{E}_{\nu_{inv}}[f(\hat{z}_k \circ \theta^k)] - \mathbb{E}_{\nu_{inv}}[f(\hat{x}_{2k})]\Big| \\
\leqslant \quad & \mathbb{E}_\nu[d(\hat{x}_{k+k}, \hat{z}_k \circ \theta^k)] + \Big|\mathbb{E}_{\rho_\nu}[f(\hat{z}_k \circ \theta^k)] - \mathbb{E}_{\rho_{inv}}[f(\hat{z}_k \circ \theta^k)]\Big| \\
& + \mathbb{E}_{\nu_{inv}}[d(\hat{x}_{k+k}, \hat{z}_k \circ \theta^k)]
\end{aligned}$$

- Conclusion $W_1(\nu\Pi^n, \nu_{inv}) \leqslant C\lambda^n$

Conclusion

Remarks

- In parallel to the study of the Markov chain $(\hat{x}_n)$ one can also study the process (X_n) valued in $\{1\dots,\ell\}$ which corresponds to the index i corresponding to the transition V_i such that $\hat{x}_{n+1} = V_i \cdot \hat{x}_n$. This process is attached to the so-called measurement records. In particular $N_n(i) = \sum \mathbf{1}_i(X_n)$ correspond to the number of occurrence of i. It represents the number of times where we have observed the result i.
- The process (X_n) is not a Markov chain but the process $(\hat{x}_n, X_n)$ is. In particular we have the following theorem concerning the invariant measure and the convergence towards this measure

Proposition

Assume ... Let ν_{inv} be the invariant measure of $(\hat{x}_n)$. Then $(\hat{x}_n, X_n)$ has a unique invariant measure denoted by $\tilde{\nu}_{inv}$ defined by

$$\mathbb{E}_{\tilde{\nu}_{inv}}[f(\hat{x}, X)] = \sum_i \int_\Omega f(\hat{x}, i) \|V_i \hat{x}\|^2 d\nu_{inv}(\hat{x})$$

Works in progress

- We get the *law of large numbers*

Theorem (Strong Law of Large Numbers)

Assume (Irr) and (Pur) hold. Assume f is continuous. Let ν be a measure over $(P(\mathbb{C}^k), \mathcal{B})$. Then,

$$\lim_{t \to \infty} \frac{1}{n} S_n(f) = \nu_{inv.}(f) \quad \mu_\nu - a.s.$$

- The above LLN generalizes the mean Cesaro result of H. Maasen and B. Kümmerer.
- The CLT for a class of α Hölder functions...
- LDP...
- Spectral study of Π.
- Open question: support of ν_{inv}.
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Thank You