On asymmetric quantum hypothesis testing

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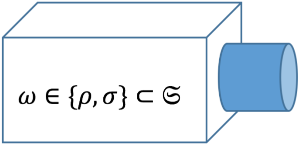
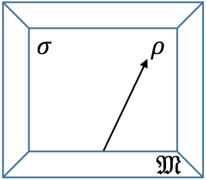
Overview

1. Motivation: the quantum Stein lemma and its refinements
2. Second-order asymptotics in Stein's lemma
3. Some examples
4. Finite sample size quantum hypothesis testing: the i.i.d. case

Framework





- Let $\mathcal{H}$ finite dimensional Hilbert space, $\rho, \sigma \in \mathcal{D}(\mathcal{H})_+$, set of states on $\mathcal{H}$
- Goal is to distinguish between:

ρ (null hypothesis) *and* σ (alternative hypothesis).

- A test is a POVM $\{T, \mathbf{1} - T\}$, $T \in \mathcal{B}(\mathcal{H})$, $0 \leq T \leq \mathbf{1}$
- Errors when guessing are therefore given by:

$$\alpha(T) = \mathrm{Tr}(\rho(\mathbf{1} - T)) \qquad \text{type I error}$$
$$\beta(T) = \mathrm{Tr}(\sigma T) \qquad \text{type II error}$$

- Asymmetric case, minimize the type II error while controlling the type I error:

$$\beta(\epsilon) = \inf_{0 \leq T \leq \mathbf{1}} \{\beta(T) |\ \alpha(T) \leq \epsilon\}.$$

Size-dependent hypothesis testing

- Let $\mathcal{H}_n$ a sequence of Hilbert spaces, $\{\rho_n\}_{n\in\mathbb{N}}$, $\{\sigma_n\}_{n\in\mathbb{N}} \in \mathcal{D}(\mathcal{H}_n)_+$.
- Examples:
  - the i.i.d. case:
  $$\rho_n = \rho^{\otimes n} \quad vs. \quad \sigma_n = \sigma^{\otimes n}.$$
  - Gibbs states on a lattice Λ_n of size $|\Lambda_n| = n^d$:
  $$\rho_n := \frac{e^{-\beta_\rho H^\rho_{\Lambda_n}}}{\mathrm{Tr}\left(e^{-\beta_\rho H^\rho_{\Lambda_n}}\right)} \quad vs. \quad \sigma_n := \frac{e^{-\beta_\sigma H^\sigma_{\Lambda_n}}}{\mathrm{Tr}\left(e^{-\beta_\sigma H^\sigma_{\Lambda_n}}\right)}.$$
  - Quasi-free fermions on a lattice etc.
- Question: Given a uniform bound ϵ on the type I errors, how quickly does $\beta_n(\epsilon)$ decrease with n?

First-order asymptotics

- First proved in the i.i.d. setting

Theorem (Quantum Stein's lemma Hiai&Petz91, Ogawa&Nagaoka00)

When $\rho_n = \rho^{\otimes n}$, *and* $\sigma_n = \sigma^{\otimes n}$,

$$-\frac{1}{n}\log\beta_n(\epsilon) \to D(\rho\|\sigma), \qquad \forall\epsilon\in(0,1), \quad \textit{where}$$

$$D(\rho\|\sigma) = \mathrm{Tr}(\rho(\log\rho - \log\sigma)) \quad \textit{Umegaki's relative entropy.}$$

$r < D(\rho\|\sigma) \Rightarrow$

$$\lim_{n\to\infty} -\frac{1}{n}\log\min\{\alpha(T_n):\ \beta(T_n) \le e^{-nr}\} := \sup_{0<\alpha<1}\frac{\alpha-1}{\alpha}[r - D_\alpha(\rho\|\sigma)],$$

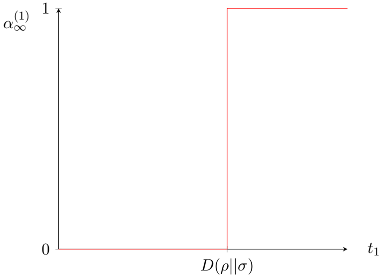
$r > D(\rho\|\sigma) \Rightarrow$

$$\lim_{n\to\infty} -\frac{1}{n}\log\max\{(1-\alpha(T_n)) : \beta(T_n) \le e^{-nr}\} = \sup_{1<\alpha}\frac{\alpha-1}{\alpha}[r - D_\alpha^*(\rho\|\sigma)], \ \textit{where}$$

$$D_\alpha(\rho\|\sigma) := \frac{1}{\alpha-1}\log\mathrm{Tr}(\rho^\alpha\sigma^{1-\alpha}) \qquad \textit{Rényi-}\alpha\textit{ divergence,}$$

$$D_\alpha^*(\rho\|\sigma) := \frac{1}{\alpha-1}\log\mathrm{Tr}(\rho^{1/2}\sigma^{\frac{1-\alpha}{\alpha}}\rho^{1/2})^\alpha \qquad \textit{sandwiched Rényi-}\alpha\textit{ divergence.}$$

Interpretation of Stein's lemma



$$\alpha_\infty^{(1)} = \inf_{0\le T_n\le \mathbf{1}}\{\limsup_n \alpha(T_n) : \; -\liminf_n \frac{1}{n}\log\beta(T_n) \ge t_1\}.$$

- Discontinuity of $\alpha_\infty^{(1)}$: manifestation of coarse-grained analysis.
- Question: can we better quantify the behaviour of the asymptotic type I error by adding a sub-exponential term for the type II error?[1]

[1] Lots of generalisations to the non i.i.d. framework: Bjelakovic&Siegmund-Schultze04, Bjelakovic,&Deuschel&Krueger&Seiler&Siegmund-Schultze&Szkola08, Hiai&Mosonyi&Ogawa08, Mosonyi&Hiai&Ogawa&Fannes08, Brandao&Plenio10, Jaksic&Ogata&Pillet&Seiringer12, etc.

Second-order asymptotics in the i.i.d. case

- Tomamichel&Hayashi13 and Li14 proved the following second-order result:

Theorem (Second-order asymptotics: the i.i.d. case Tomamichel&Hayashi13, Li14)

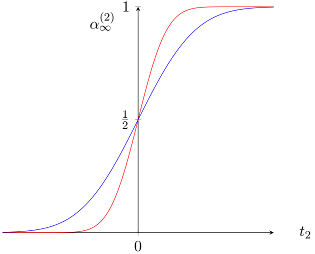
When $\rho_n = \rho^{\otimes n}$ *and* $\sigma_n = \sigma^{\otimes n}$,

$$-\log \beta_n(\epsilon) = nD(\rho\|\sigma) + \sqrt{n}s_1(\epsilon) + \mathcal{O}(\log n), \quad s_1(\epsilon) := \sqrt{V(\rho\|\sigma)}\Phi^{-1}(\epsilon),$$

where

$$V(\rho\|\sigma) = \mathrm{Tr}(\rho(\log\rho - \log\sigma)^2) - D(\rho\|\sigma)^2 \quad \textit{quantum information variance},$$

and Φ *the cumulative distribution function of law* $\mathcal{N}(0,1)$.



$$\alpha_\infty^{(2)}(t_2) = \inf_{0\le T_n \le \mathbf{1}} \left\{ \limsup_n \alpha(T_n) \,\middle|\, -\liminf_n \frac{1}{\sqrt{n}}\left(\log\beta_n(T_n) + n\,D(\rho\|\sigma)\right) \ge t_2 \right\} = \Phi(t_2/\sqrt{V(\rho\|\sigma)}).$$

Main result 1: Second-order asymptotics in the non-i.i.d. scenario

- For a given strictly increasing sequence of weights w_n, **under some conditions** (see later), the *quantum information variance rate*

$$v(\{\rho_n\},\{\sigma_n\}) = \lim_{n\to\infty} \frac{1}{w_n} V(\rho_n\|\sigma_n)$$

is well-defined, and:

Theorem (DPR16)

Fix $\epsilon \in (0,1)$. *Define*

$$t_2^*(\epsilon) = \sqrt{v(\{\rho_n\},\{\sigma_n\})}\,\Phi^{-1}(\epsilon).$$

Then

- *Optimality:* $\forall\ t_2 > t_2^*(\epsilon)$ *there exists a function* $f_{t_2}(x) \underset{+\infty}{=} o(\sqrt{x})$ *such that* $\forall n \in \mathbb{N}$ *and any sequence of tests* T_n*:*

$$-\log\beta(T_n) \le D(\rho_n||\sigma_n) + \sqrt{w_n}\,t_2 + f_{t_2}(w_n), \tag{1}$$

- *Achievability:* $\forall\ t_2 < t_2^*(\epsilon)$, $\forall f(x) \underset{+\infty}{=} o(\sqrt{x})$, *there exists a sequence of test* T_n *such that for* n *large enough:*

$$-\log\beta_n(T_n) \ge D(\rho_n||\sigma_n) + \sqrt{w_n}\,t_2 + f(w_n). \tag{2}$$

- The i.i.d. case of Tomamichel&Hayashi13 and Li14 follows (Bryc $\leftarrow$ CLT). The Berry-Esseen theorem even provides information on third order:

$$-\log\beta_n(\epsilon) = nD(\rho\|\sigma) + \sqrt{nV(\rho\|\sigma)}\Phi^{-1}(\epsilon) + \mathcal{O}(\log n),$$

Key tools 1: the relative modular operator

- The relative modular operator is a generalisation of the Radon-Nikodym derivative for general von Neuman algebras. In finite dimensions, for two faithful states, it reduces to:

$$\Delta_{\rho|\sigma} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}), \qquad A \mapsto \rho A \sigma^{-1}.$$

- $\Delta_{\rho|\sigma}$ is a positive operator, admits a spectral decomposition.
- Quantities of relevance can be rewritten in terms of $\Delta_{\rho|\sigma}$:

$$D(\rho\|\sigma) = \langle \rho^{1/2}, \log(\Delta_{\rho|\sigma})(\rho^{1/2})\rangle, \qquad \text{where } \langle A, B\rangle = \mathrm{Tr}(A^* B).$$
$$\Psi_s(\rho|\sigma) := \log \mathrm{Tr}(\rho^{1+s}\sigma^{-s}) = \log\langle \rho^{1/2}, \Delta^s_{\rho|\sigma}(\rho^{1/2})\rangle$$

- Define X to be the classical random variable taking values on $\mathrm{spec}(\log \Delta_{\rho|\sigma})$ such that for any measurable $f : \mathrm{spec}(\log \Delta_{\rho|\sigma}) \to \mathbb{R}$,

$$\langle \rho^{1/2}, f(\log \Delta_{\rho|\sigma})(\rho^{1/2})\rangle \equiv \mathbb{E}[f(X)].$$

- $f : x \mapsto x \Rightarrow \mathbb{E}[X] = D(\rho\|\sigma)$,
  $f : x \mapsto e^{sx} \Rightarrow \log \mathbb{E}[e^{sX}] = \log\langle \rho^{1/2}, e^{s \log \Delta_{\rho|\sigma}}(\rho^{1/2})\rangle \equiv \Psi_s(\rho|\sigma)$, the cumulant-generating function.

Key tools 2: Bryc's theorem

- The following theorem is a non i.i.d. generalization of the Central limit theorem:

Theorem (Bryc's theorem Bryc93)

Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of random variables, and $(w_n)_{n\in\mathbb{N}}$ an increasing sequence of weights. If there exists $r > 0$ such that:

- $\forall n \in \mathbb{N}$, $H_n(z) := \log \mathbb{E}\left[e^{zX_n}\right]$ *is analytic in the complex open ball* $B_{\mathbb{C}}(0, r)$,
- $H(x) = \lim_{n\to\infty} \frac{1}{w_n} H_n(x)$ *exists* $\forall x \in (-r, r)$,
- $\sup_{n\in\mathbb{N}} \sup_{z\in B_{\mathbb{C}}(0,r)} \frac{1}{w_n} |H_n(z)| < +\infty$,

then H is analytic on $B_{\mathbb{C}}(0, r)$ and

$$\frac{X_n - H_n'(0)}{\sqrt{w_n}} \xrightarrow{\mathrm{d}}_{n\to\infty} \mathcal{N}(0, H''(0)),$$

The working condition

- Take X_n associated to $\log \Delta_{\rho_n|\sigma_n}$, $\rho_n^{1/2} \Rightarrow H_n(z) = \Psi_z(\rho_n|\sigma_n)$.
- The conditions of Bryc's theorem translate into:

Condition

Let $\{w_n\}$ an increasing sequence of weights. Assume there exists $r > 0$ such that

- $\forall n \in \mathbb{N}$, $z \to \Psi_z(\rho_n|\sigma_n)$ *is analytic in the complex open ball* $B_{\mathbb{C}}(0, r)$,
- $H(x) = \lim_{n\to\infty} \frac{1}{w_n} \Psi_z(\rho_n|\sigma_n)$ *exists* $\forall x \in (-r, r)$,
- $\sup_{n\in\mathbb{N}} \sup_{z\in B_{\mathbb{C}}(0,r)} \frac{1}{w_n} |\Psi_z(\rho_n|\sigma_n)| < +\infty$,

- Here $H_n'(0) = D(\rho_n\|\sigma_n)$,
- By Bryc's theorem, the quantum information variance rate

$$v(\{\rho_n\}, \{\sigma_n\}) = \lim_{n\to\infty} \frac{1}{w_n} V(\rho_n\|\sigma_n) \equiv H''(0)$$

is well-defined.

Proof of achievability (2)

- We will need the following crucial technical lemma:

Lemma (Li14, DPR16)

Let $\rho, \sigma \in \mathcal{D}_+(\mathcal{H})$. For all $L > 0$ there exists a test T such that

$$\mathrm{Tr}(\rho(\mathbf{1} - T)) \leq \langle \rho^{1/2}, P_{(0,L)}(\Delta_{\rho|\sigma})(\rho^{1/2}) \rangle \qquad \textit{and} \qquad \mathrm{Tr}(\sigma T) \leq L^{-1}. \tag{3}$$

- Let $L_n := \exp(D(\rho_n \| \sigma_n) + \sqrt{w_n} t_2 + f(w_n))$, $f(x) = \circ(\sqrt{x})$.

$$(3) \quad \Rightarrow \quad -\log \beta(T_n) \geq D(\rho_n || \sigma_n) + \sqrt{w_n}\, t_2 + f(w_n)$$

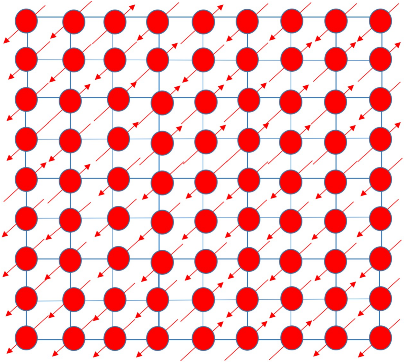
$$\begin{aligned}\alpha(T_n) &\leq \langle {\rho_n}^{1/2}, P_{(0,L_n)}(\Delta_{\rho_n|\sigma_n})\, {\rho_n}^{1/2} \rangle, \\ &= \langle {\rho_n}^{1/2}, P_{(-\infty, \log L_n)}(\log \Delta_{\rho_n|\sigma_n})\, {\rho_n}^{1/2} \rangle. \\ &= \mathbb{P}(X_n \leq \log L_n) = \mathbb{P}\left(\frac{X_n - D(\rho_n || \sigma_n)}{\sqrt{w_n}} \leq t_2 + \frac{f(w_n)}{\sqrt{w_n}} \right).\end{aligned}$$

- Bryc's theorem $\Rightarrow$ RHS converges to $\Phi(t_2/\sqrt{v(\{\rho_n\}, \{\sigma_n\})}) < \varepsilon \Rightarrow$ for n large enough,

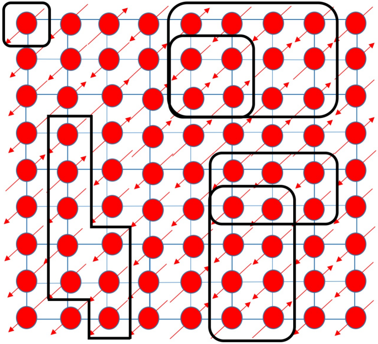
$$\alpha(T_n) \leq \varepsilon.$$

- Optimality (1) follows similarly from a lower bound on the total error provided by Jaksic&Ogata&Pillet&Seiringer12.

Quantum spin systems



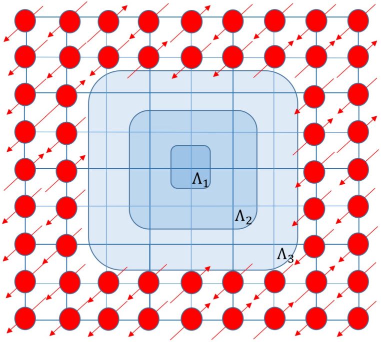
- Lattice $\mathbb{Z}^d$.
- System prepared at each site: $\forall x, \mathcal{H}_x = \mathcal{H}$.
- Example: particle (spin $\pm\frac{1}{2}$): $\mathcal{H}_x = \mathbb{C}^2$.



- $\forall X \subset \mathbb{Z}^d, \mathcal{H}_X := \otimes_{x \in X} \mathcal{H}_x$,
- **Interaction** between sites: $\Phi : X \mapsto \Phi_X \in \mathcal{B}_{sa}(\mathcal{H}_X)$.
- Φ **Translation invariant**: invariant on sets of same shape.
- Φ **Finite range**: $\exists R > 0 : \mathrm{diam}(X) > R \Rightarrow \Phi_X = 0$.
- Φ induces dynamics, with equilibrium states (**Gibbs states**):

$$H_X^{\Phi} := \sum_{Y \subset X} \Phi_Y$$

$$\rho_X^{\Phi,\beta} := \frac{\mathrm{e}^{-\beta H_X^{\Phi}}}{\mathrm{Tr}(\mathrm{e}^{-\beta H_X^{\Phi}})}.$$



- $\Lambda_n := \{-n, ..., n\}^d$.
- Translation invariant, of finite range interactions Φ, Ψ.
- Suppose given one of two states

$$\rho_n := \rho_{\Lambda_n}^{\Phi,\beta_1} \text{ or } \sigma_n := \rho_{\Lambda_n}^{\Psi,\beta_2}$$

- Tests performed on $\mathcal{H}_{\Lambda_n}$

Proposition (DPR16)

For β_1, β_2 small enough, ρ_n and σ_n satisfy the conditions for second order asymptotics w.r.t. $w_n = |\Lambda_n|$.

Free lattice fermions

- One-particle Hilbert space $\mathfrak{h} := l^2(\mathbb{Z}^d)$, standard basis $\{e_i : \ i \in \mathbb{Z}^d\}$ (particle localized at site i).
- Construct the Fock space $\mathcal{F}(\mathfrak{h}) = \bigoplus_{n\in\mathbb{N}} \bigwedge^n \mathfrak{h}$, where

$$x_1 \wedge \ldots \wedge x_n = \frac{1}{\sqrt{n!}} \sum_{\sigma\in S_n} \mathrm{sgn}(\sigma) x_{\sigma(1)} \otimes \ldots \otimes x_{\sigma(n)}.$$

- Algebra of observables: C^* algebra generated by creation and anihilation operators on the Fock space, obeying the *canonical anti-commutation relations*:

$$a(x)a(y) + a(y)a(x) = 0, \quad a(x)a^*(y) + a^*(y)a(x) = \langle x, y\rangle \mathbf{1}, \ x, y \in \mathfrak{h}.$$

- $a^*(y)$: unique linear extension $a^*(y) : \mathcal{F}(\mathfrak{h}) \to \mathcal{F}(\mathfrak{h})$ of

$$a^*(y)x_1 \wedge \ldots \wedge x_n = y \wedge x_1 \wedge \ldots \wedge x_n, \qquad a(y) = (a^*(y))^*$$

- Let $H \in \mathcal{B}(\mathfrak{h})$ be a one-particle Hamiltonian, define $\hat{H} := d\Gamma(H)$ the differential second quantization on $\mathcal{F}(\mathfrak{h})$, closure of

$$H_n(x_1 \wedge ... \wedge x_n) = P_- \left(\sum_{k=1}^{n} x_1 \otimes ... \otimes Hx_k \otimes ... \otimes x_n \right),$$

  where P_- is the projection onto the antisymmetric subspace.

- Define the Gibbs state:

$$\rho_\beta^H = \frac{\mathrm{e}^{-\beta \hat{H}}}{\mathrm{Tr}\left(\mathrm{e}^{-\beta \hat{H}}\right)}$$

- Gibbs states are expressed as linear forms on $CAR(\mathfrak{h})$ as follows:

$$\mathrm{Tr}(\rho_\beta^H a^*(x_1)...a^*(x_n)a(y_m)...a(y_1)) = \delta_{mn} \det(\langle y_i, Qx_i \rangle)_{i,j}, \qquad Q := \frac{\mathrm{e}^{-\beta H}}{1 + \mathrm{e}^{-\beta H}}.$$

- ρ_β^H is called the quasi-free state with *symbol* Q.
- Define the *shift operators* $T_j : e_i \to e_{i+j}$, $i, j \in \mathbb{Z}^d$. A symbol which commutes with all shift operators is said to be *shift invariant.*

- Assume now that we only have access to part of the lattice: denote

$$\Lambda_n := \{0, \ldots, n-1\}^d \subset \mathbb{Z}^d, \qquad \mathfrak{h}_n = l^2(\Lambda_n) \subset \mathfrak{h}.$$

- Assume given two shift invariant symbols $\delta < Q, R < 1 - \delta$.
- Denote $Q_n = P_n Q P_n$, $R_n = P_n R P_n$, where P_n orthogonal projection onto $\mathfrak{h}_n$.
- The sequences of states that we want to distinguish between are then:

$\{\rho_n\}$ associated with symbols $\{Q_n\}$ *vs.* $\{\sigma_n\}$ associated with symbols $\{R_n\}$.

- Dierckx&Fannes&Pogorzelska08: ρ_n, σ_n can be written as:

$$\rho_n = \det(\mathbf{1} - Q_n) \bigoplus_{k=0}^{\dim \mathfrak{h}_n} \bigwedge^k \frac{Q_n}{\mathbf{1} - Q_n} \qquad \sigma_n = \det(\mathbf{1} - R_n) \bigoplus_{k=0}^{\dim \mathfrak{h}_n} \bigwedge^k \frac{R_n}{\mathbf{1} - R_n}.$$

Proposition (DPR16)

ρ_n, and σ_n satisfy the conditions for second order asymptotics w.r.t. $w_n = |\Lambda_n|$.

Finite sample size bounds of Audenaert&Mosonyi&Verstraete12

- Practical situations: finitely many copies available.

Theorem (Audenaert&Mosonyi&Verstraete12)

For $\rho_n := \rho^{\otimes n}$ *and* $\sigma_n := \sigma^{\otimes n}$,

$$-D(\rho\|\sigma) - \frac{f(\epsilon)}{\sqrt{n}} \leq \frac{1}{n}\log\beta_n(\epsilon) \leq -D(\rho\|\sigma) + \frac{g(\epsilon)}{\sqrt{n}},$$

where

$$f(\epsilon) = 4\sqrt{2}\log\eta\log(1-\epsilon)^{-1}, \qquad g(\epsilon) = 4\sqrt{2}\log\eta\log\epsilon^{-1},$$

and $\eta := 1 + \mathrm{e}^{1/2 D_{3/2}(\rho\|\sigma)} + \mathrm{e}^{-1/2 D_{1/2}(\rho\|\sigma)}$.

- The 'second order' parts of the bounds scale as $\log\epsilon^{-1}$, to compare with $\Phi^{-1}(\epsilon)$ in the asymptotic case $\Rightarrow$ not tight.
- A better upper bound can be found by means of (non-commutative) martingale concentration inequalities.

Non-commutative martingales

- A *noncommutative probability space* is a couple $(\mathcal{M}, \tau)$, where $\mathcal{M}$ is a von Neumann algebra, and τ a normal tracial state on $\mathcal{M}$. Let $\mathcal{N}$ be a von Neumann subalgebra of $\mathcal{M}$. Then there exists a unique map $\mathbb{E}[.|\mathcal{N}] : \mathcal{M} \to \mathcal{N}$, called *conditional expectation*, such that

$$\mathbb{E}[\mathbf{1}|\mathcal{N}] = \mathbf{1},$$
$$\mathbb{E}[AXB|\mathcal{N}] = A\mathbb{E}[X|\mathcal{N}]B, \quad A, B \in \mathcal{N},\ X \in \mathcal{M},$$
$$\mathbb{E}_*[\tau|\mathcal{N}] = \tau,$$

- A *noncommutative* filtration of $\mathcal{M}$ is an increasing sequence $\{\mathcal{M}_j\}_{1 \le j \le n}$ of von Neumann subalgebras of $\mathcal{M}$.
- A *martingale* is a sequence of noncommutative random variables $\{X_j\}_{1 \le j \le n} \in \mathcal{M}^n$ such that for each j,

$$X_j \in \mathcal{M}_j,$$
$$\tau(|X_j|) < \infty \qquad \text{(integrability)},$$
$$\mathbb{E}[X_{j+1}|\mathcal{M}_j] = X_j \qquad \text{(martingale property)}.$$

A noncommutative martingale concentration inequality

Theorem (Noncommutative Azuma inequality, Sadeghi&Moslehian14)

Let $\{X_j, \mathcal{M}_j\}_{1\le j\le n}$ *be a self-adjoint martingale. If* $-d_j \le X_{j+1} - X_j \le d_j$ $(d_j > 0)$,

$$\tau(\mathbf{1}_{[\alpha,\infty)}(X_n)) \le \exp\left(\frac{-\alpha^2}{2\sum_{j=1}^n d_j^2}\right), \qquad \alpha > 0.$$

- Idea (borrowed from Sason11):
  - Take $\rho_n = \tilde\rho_1 \otimes \ldots \otimes \tilde\rho_n$, and $\sigma_n = \tilde\sigma_1 \otimes \ldots \otimes \tilde\sigma_n$. Then

$$\Delta_{\rho_n|\sigma_n} = \bigotimes_{j=1}^n \Delta_{\tilde\rho_j|\tilde\sigma_j} \Rightarrow \log \Delta_{\rho_n|\sigma_n} = \sum_{j=0}^{n-1} \mathrm{id}^{\otimes j} \otimes \log \Delta_{\tilde\rho_j|\tilde\sigma_j} \otimes \mathrm{id}^{\otimes n-j-1}$$

  - Define algebras $\mathcal{M}_k$ generated by $\mathrm{id}^i \otimes \log \Delta_{\rho_i|\sigma_i} \otimes \mathrm{id}^{n-i-1}$, $i \le k$.
  - $\langle \rho_n^{1/2}, (.)\ \rho_n^{1/2}\rangle$ is a tracial state on $\mathcal{M}_n$, and $(\log \Delta_{\rho_k|\sigma_k} - D(\rho_k\|\sigma_k), \mathcal{M}_k)$ is a martingale, where $\mathbb{E}[.|\mathcal{M}_k] := \langle \tilde\rho_{k+1} \otimes \ldots \otimes \tilde\rho_n, (.)\tilde\rho_{k+1} \otimes \ldots \otimes \tilde\rho_n\rangle$.

- $\{\log \Delta_{\rho_k|\sigma_k} - D(\rho_k\|\sigma_k)\}_{1\le k\le n}$ is a self-adjoint martingale, hence by NC Azuma:

$$\langle \rho_n^{1/2}, \mathbf{1}_{(r,\infty)}(\log(\Delta_{\rho_n|\sigma_n}) - D(\rho_n|\sigma_n))(\rho_n^{1/2})\rangle \le \mathrm{e}^{-\frac{r^2}{2\sum_j d_j^2}}.$$

- Recall: $\forall n, L_n, \exists T_n$:

$$\alpha(T_n) \le \langle \rho_n^{1/2}, P_{(0,L_n)}(\Delta_{\rho_n|\sigma_n})(\rho_n^{1/2})\rangle \quad \text{and} \quad \beta(T_n) \le L_n^{-1}.$$

Hence,

$$\begin{aligned}
\alpha(T_n) &\le \langle \rho_n^{1/2}, P_{(-\infty,\log L_n)}(\log \Delta_{\rho_n|\sigma_n})(\rho_n^{1/2})\rangle \le \mathrm{e}^{-\frac{(\log L_n - D(\rho_n\|\sigma_n))^2}{2\sum_j d_j^2}} \equiv \epsilon \\
&\Rightarrow \log L_n \equiv \sqrt{2\sum_j d_j^2 \ln \epsilon^{-1}} + D(\rho_n\|\sigma_n) \\
&\Rightarrow \beta_n(\epsilon) \le \beta(T_n) \le \mathrm{e}^{-D(\rho_n\|\sigma_n) - \sqrt{2\sum_j d_j^2 \log \epsilon^{-1}}}.
\end{aligned}$$

- Remember Audenaert&Mosonyi&Verstraete12 bound:

$$\log \beta_n(\epsilon) \le -nD(\rho\|\sigma) + \sqrt{n} g(\epsilon), \quad g(\epsilon) := K_{\rho,\sigma} \log \epsilon^{-1}.$$

Theorem (DR16)

Suppose given states of the form

$$\rho_n = \bigotimes_{j=1}^{n} \tilde{\rho}_j \qquad \text{and} \qquad \sigma_n = \bigotimes_{j=1}^{n} \tilde{\sigma}_j,$$

where for each j, $\tilde{\rho}_j, \tilde{\sigma}_j \in \mathcal{D}(\mathcal{H}_j)_+$. Then for each $n \in \mathbb{N}$:

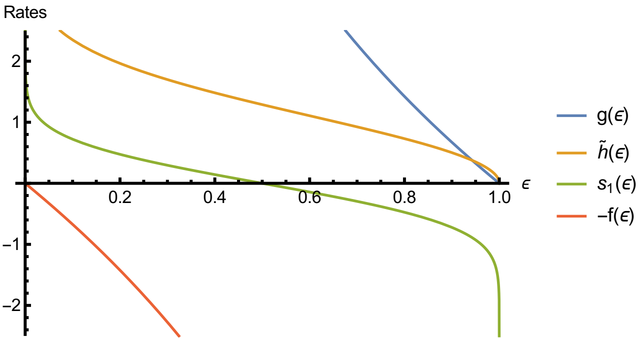
$$\log \beta_n(\epsilon) \le -\sum_{j=1}^{n} D(\tilde{\rho}_j\|\tilde{\sigma}_j) + \sqrt{2\log \epsilon^{-1} \sum_{k=1}^{n} d_k^2}, \quad d_k := \| \log \Delta_{\tilde{\rho}_k|\tilde{\sigma}_k} - D(\tilde{\rho}_k\|\tilde{\sigma}_k)\|_\infty.$$

In the i.i.d. case ($\rho_n = \rho^{\otimes n}$ vs. $\sigma_n = \sigma^{\otimes n}$),

$$\log \beta_n(\epsilon) \le -nD(\rho\|\sigma) + \sqrt{n}\tilde{h}(\epsilon), \quad \tilde{h}(\epsilon) := \sqrt{2\log \epsilon^{-1}}\, d_1.$$

Comparison with Audenaert&Mosonyi&Verstraete12

- Our error scales as: $\sim \sqrt{\log \epsilon^{-1}}$, as opposed to $\sim \log \epsilon^{-1}$.
- We are getting closer to the asymptotic behavior in $\sim \Phi^{-1}(\epsilon)$.



Summary and open questions

- We studied second order asymptotics as well as finite sample size hypothesis testing in the asymmetric scenario.
- We proved a theorem providing second order asymptotics for a large class of quantum systems, including the i.i.d. scenario of Tomamichel&Hayashi13 and Li14, as well as states of quantum spin systems and free fermions on a lattice.
- We found finite sample size bounds on the optimal type II error in the i.i.d. scenario using noncommutative martingale concentration inequalities.
- Open question: Does the martingale approach give bounds in physically relevant non i.i.d. examples?

- Thank you for your attention!

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