Quantum statistics:

estimation and metrology for quantum open systems

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M.G., J. Kiukas, arXiv:1601.04355

M.G., J. Kiukas, Commun. Math. Phys. (2015) arXiv:1402.3535

K. Macieszczak, M.G. I. Lesanovsky, J. P. Garrahan, Phys. Rev. A (2016) arXiv:1411.3914

K. Macieszczak, M.G. I. Lesanovsky, J. P. Garrahan, Phys. Rev. Lett. (2016) arXiv:1512.05801

EPSRC

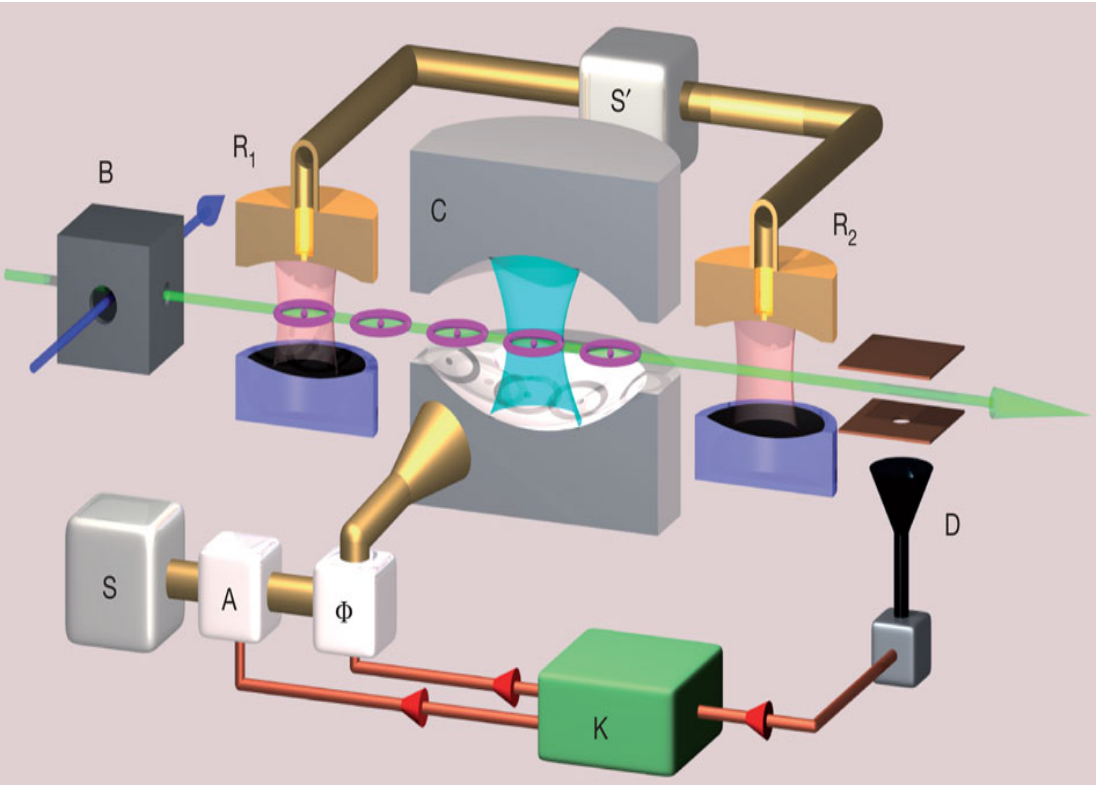
Quantum trajectories, parameter and state estimation
Toulouse 2017

The University of Nottingham

Quantum-classical interface is stochastic ⇒ Q. Engineering needs Statistics

- Problem: Estimation of dynamical parameters (system identification)
  - Controlling quantum systems is challenging due to high sensitivity to "noise"
  - Use stochastic feedback control based on output measurement process
  - Estimate interaction parameters from the measurement process

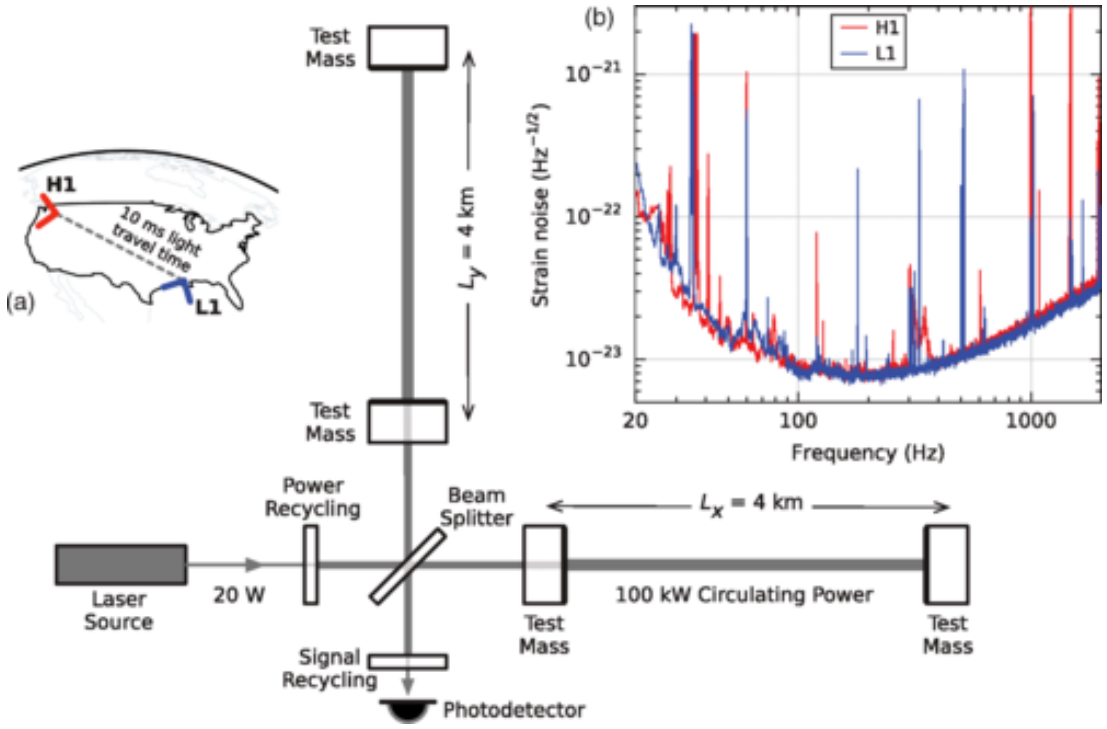
Watt's steam engine governor



Feedback control of cavity state in the atom maser

[C. Sayrin *et al*, Nature 2011]

Quantum Technology: metrology and control

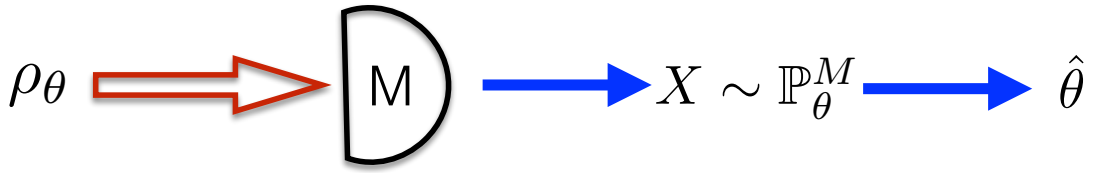


LIGO: uses quantum coherence to detect changes in distance of order 10^{-18} m

Outline

- Quantum parameter estimation
  - Standard vs Heisenberg scaling
  - Discrete time Markov dynamics
- Parameter estimation for discrete time quantum Markov chains
  - CLT for output measurements average statistics
  - Classical Fisher information for average statistics
  - Quantum Fisher information and LAN for output
  - Output identifiable parameters
- Estimation and metrology in continuous time
  - atom maser and dynamical phase transitions
  - metrology at a dynamical phase transition
  - metrology near a dynamical phase transition

Quantum parameter estimation



- Estimation problem: estimate θ by performing a measurement M on system in state ρ_θ
- What is quantum about this ?
  - fixed measurement: "classical stats" problem with special probabilistic structure
  - "optimal" measurement: need to understand structure of quantum statistical model
  - quantum enhance precision when θ is encoded with "sensitive states"
- Classical and quantum Cramér-Rao bounds[1]: if $\hat{\theta}$ is unbiased

$$\mathbb{E}\left[(\hat{\theta}-\theta)^T \cdot (\hat{\theta}-\theta)\right] \geq I^M(\theta)^{-1} \geq F(\theta)^{-1}$$

Classical Fisher info — Quantum Fisher info

[1] A. Holevo. Probabilistic and Statistical Aspects of Quantum Theory 1982; S. L. Braunstein and C. M. Caves, P.R.L. 1994

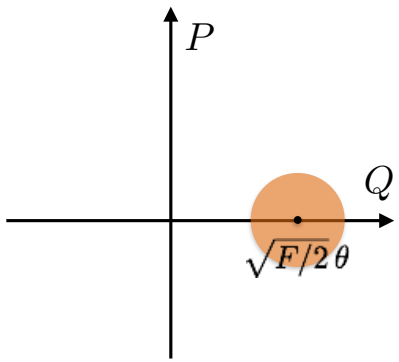
Pure states models

- one parameter pure state rotation model: $|\psi_\theta\rangle := e^{-i\theta G}|\psi\rangle, \qquad \langle\psi|G|\psi\rangle = 0$

- Quantum Fisher information:

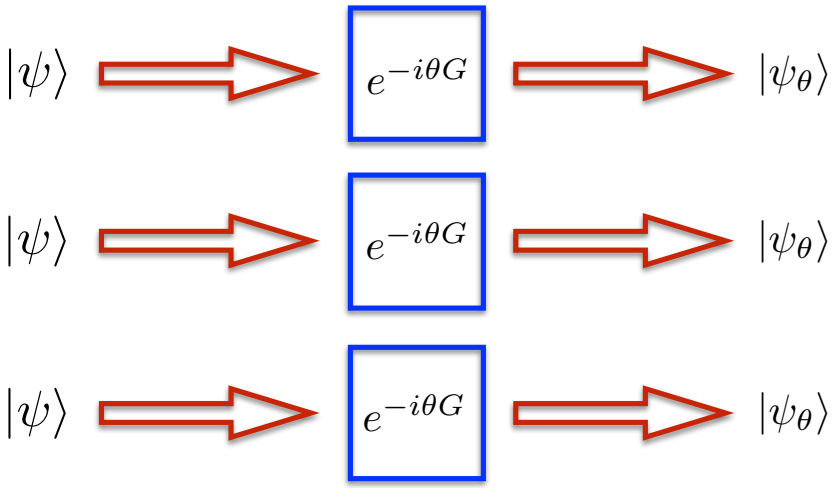
$$F(\theta) \quad = \quad 4\left\|\frac{d\psi_\theta}{d\theta}\right\|^2 = 4\mathrm{Var}_\psi(G) = 4\left\langle\psi\,\middle|\,G^2\,\middle|\,\psi\right\rangle$$

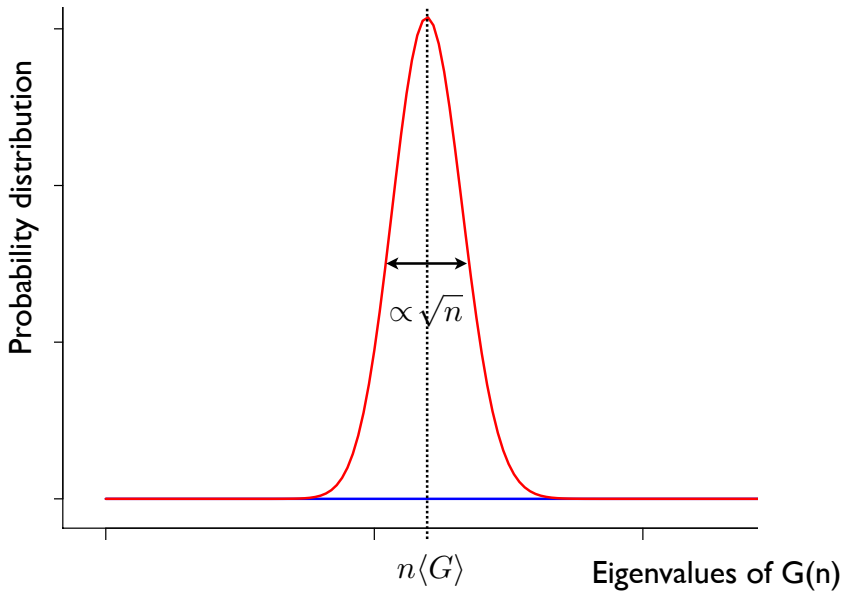
- Quantum Gaussian shift model: CV system $[\mathbf{Q},\mathbf{P}] = i\mathbf{1}$
  - $\left|\sqrt{F/2}\,\theta\right\rangle$ coherent state with mean $\left(\sqrt{F/2}\,\theta, 0\right)$
  - quantum Fisher information $= 4\mathrm{Var}\left(\sqrt{F/2}\mathbf{P}\right) = F$
  - QFI achievable by measuring $\mathbf{Q}$



- 2D quantum Gaussian shift model
  - incompatibility of $\mathbf{P}$ and $\mathbf{Q} \Rightarrow F$ not achievable
  - measurement achieving optimal MSE is linear

Standard scaling as Central Limit behaviour





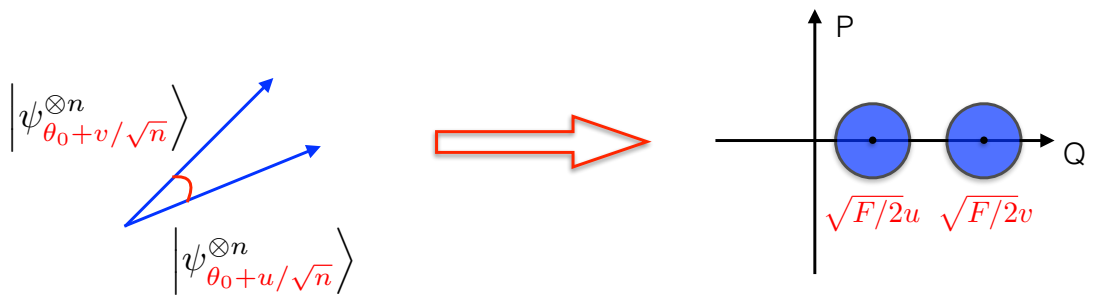
- total generator $G(n) := G^{(1)} + \cdots + G^{(n)}$
- Standard scaling: quantum Fisher information scales linearly in n

$$F_\theta(n) = 4\mathrm{Var}(G(n)) = 4n\mathrm{Var}(G)$$

Convergence to Gaussian model for i.i.d. ensembles

- Quantum data: ensemble of n identically prepared systems

$$|\psi_\theta\rangle^{\otimes n} := \left(e^{i\theta G}|\psi\rangle\right)^{\otimes n}, \qquad \langle\psi|G|\psi\rangle = 0$$



- Local asymptotic normality (Gaussian approximation):

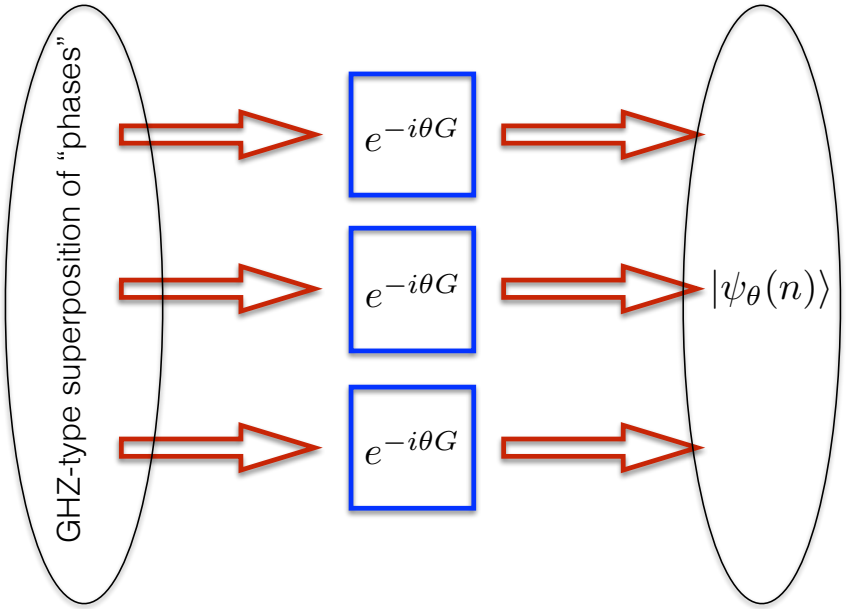
  In an "uncertainty neighbourhood" of size $n^{-1/2}$ around θ_0, the overlaps of joint states are approximately equal to those of a Gaussian model with QFI = F

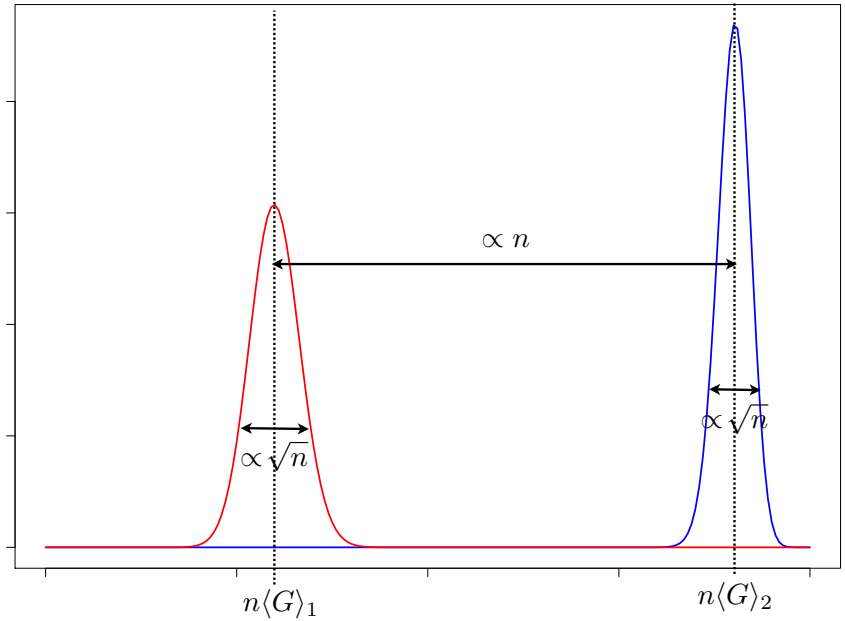
$$\left\langle \psi^{\otimes n}_{\theta_0+u/\sqrt{n}} \middle| \psi^{\otimes n}_{\theta_0+v/\sqrt{n}} \right\rangle = \left\langle \psi | e^{i(u-v)G/\sqrt{n}} \middle| \psi \right\rangle^n \longrightarrow e^{(u-v)^2 F/8} = \left\langle \sqrt{F/2}\, u \middle| \sqrt{F/2}\, v \right\rangle$$

- General LAN for mixed states & multi-dimensional models[2]

[2] J. Kahn and MG, Commun. Math. Phys. 2009

Heisenberg scaling and "phase transitions"





- Output: GHZ-type superposition of "Gaussian phases"

$$|\psi_\theta(n)\rangle = |0\rangle^{\otimes n} + e^{-in\theta}|1\rangle^{\otimes n}$$

- Heisenberg scaling: quantum Fisher information scales quadratically in n

$$F_\theta(n) = 4\mathrm{Var}(G(n)) \propto n^2$$

- Bimodal distribution of $G(n)$ reminiscent of phase transitions[3]

[3] P. Zanardi et al., Phys. Rev. A 78, 042105 (2008)

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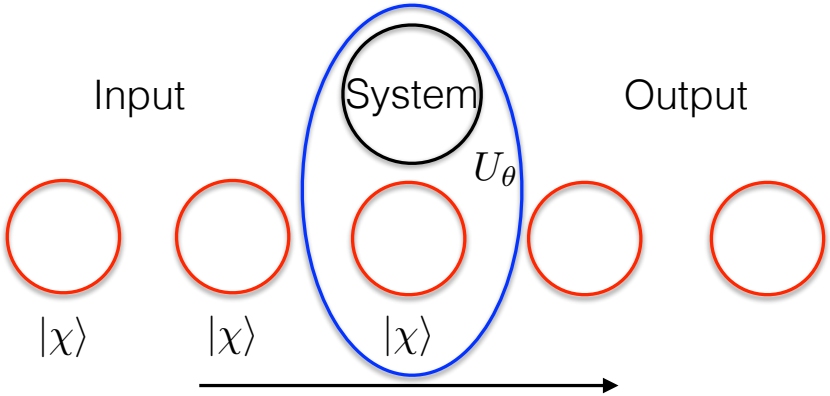
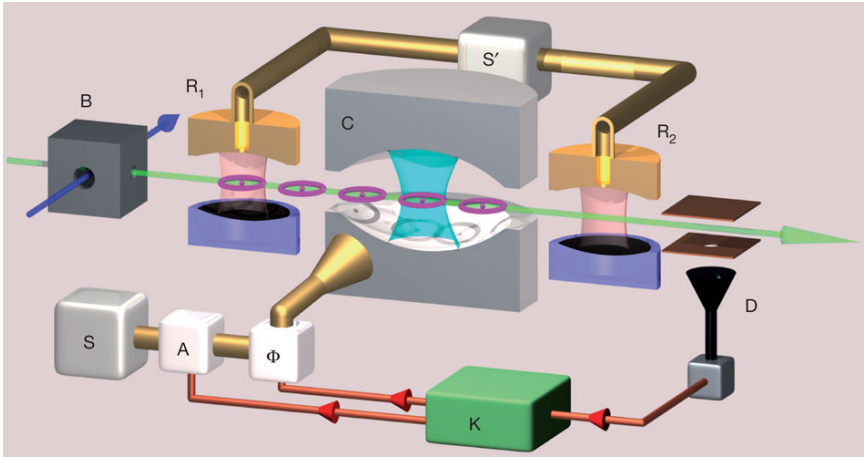
Discrete Markov dynamics = channels with memory



Successive interactions with (memory) system via unitary U_θ



Feedback control of cavity state in the atom maser
[C. Sayrin *et al*, Nature 2011]

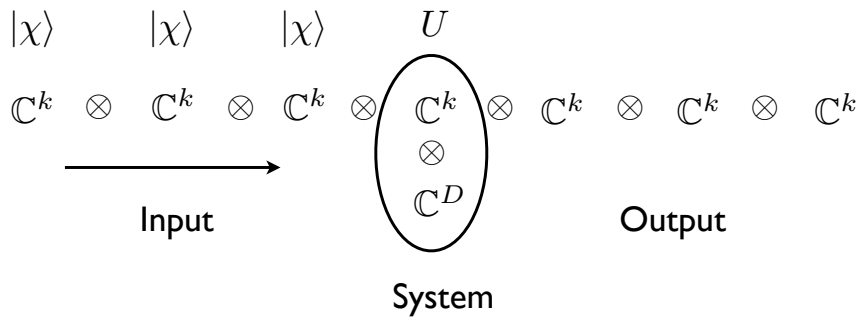
- Setup generalises both I.I.D. and GHZ examples depending on choice of U_θ
- Matrix product system-output state

$$|\Psi_\theta^{s+o}(n)\rangle = \sum_{i_1,\ldots,i_n} K_\theta^{i_n} \ldots K_\theta^{i_1} |\chi\rangle \otimes |i_n\rangle \otimes \cdots \otimes |i_1\rangle, \qquad K_\theta^i = \langle i|U_\theta|\psi\rangle$$

- System identification problem[4]: estimate θ by measuring the output state

[4] M.G., J. Kiukas, Commun. Math. Phys. 2015, M. Cramer *et al*, Nat. Commun. 2010

Quantum Markov dynamics



- Dynamics determined by isometry $V : \mathbb{C}^D \to \mathbb{C}^D \otimes \mathbb{C}^k$

$$V|\psi\rangle := U|\psi \otimes \chi\rangle = \sum_i K_i|\psi\rangle \otimes |i\rangle$$

- System-output state after n steps is of matrix product form[5]

$$|\Psi(n)\rangle = U(n)|\chi\rangle \otimes |\psi\rangle^{\otimes n} = \sum_{i_1,\ldots,i_n} K_{i_n} \ldots K_{i_1}|\chi\rangle \otimes |i_n\rangle \otimes \cdots \otimes |i_1\rangle$$

- Reduced system evolution given by transition operator $T : M(\mathbb{C}^D) \to M(\mathbb{C}^D)$

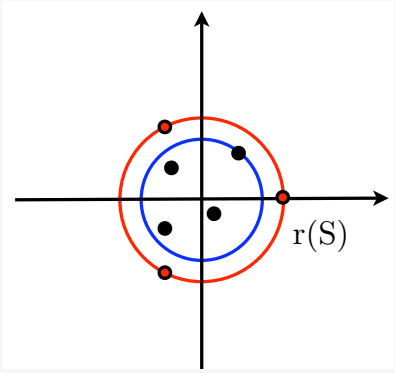
$$\begin{aligned} \rho(n) &= \mathrm{Tr}_{\mathrm{out}}(|\Psi(n)\rangle\langle\Psi(n)|) = T^n(\rho_{\mathrm{in}}), \qquad \rho_{\mathrm{in}} = |\chi\rangle\langle\chi| \\ T(\rho) &= \sum_{i=1}^{k} K_i \rho K_i^\dagger \end{aligned}$$

[5] M. Fannes, B. Nachtergale and R. Werner, 1992; D. Perez-Garcia, F. Verstraete, M. Wolf and I. Cirac, 2007

Quantum Perron-Frobenius Theorem[6]

Theorem (Quantum Perron-Frobenius Theorem)

- If T is an irreducible CP-TP map (no invariant subspaces)
  - spectral radius $r(T) = 1$ is a non-degenerate eigenvalue of T
  - unique, strictly positive stationary state: $T(\rho_{ss}) = \rho_{ss}$
  - the eigenvalues on the unit circle form a group



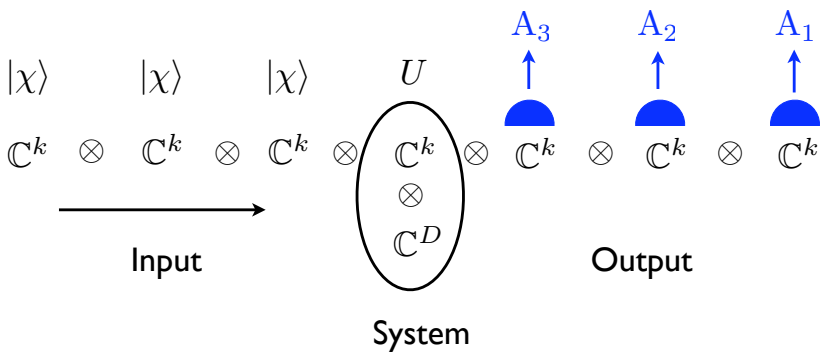
- If T is primitive (irreducible and aperiodic) then
  - $|\lambda| < 1$ for all remaining eigenvalues
  - convergence to stationary state

$$\lim_{n\to\infty} T^n(\sigma) = \rho_{ss}$$

Key observation: if T_ϵ is a small perturbation of primitive $T \Rightarrow$ dominant eigenvalue λ_ϵ varies smoothly and determines the asymptotics

[6] D. E. Evans and R. Hoegh-Krohn, J. London Math. Soc 1978; M. Sanz, et al, IEEE Trans. Inform. Th., 2010

Quantum Markov chains: sequential output measurements



- Observable $\mathbf{A} = \sum_i a_i |i\rangle\langle i|$ measured on each unit $\longrightarrow$ outcomes $A_1, A_2, \ldots$
- Statistic: time (empirical) average $S_n(A) = \frac{1}{n}\sum_{i=1}^n A_i$
- Moment generating function

$$\phi(s) := \mathbb{E}\left(e^{snS_n(A)}\right) = \mathrm{Tr}(T_s^n(\rho_{\mathrm{in}}))$$

- Deformed transition operator $T_s : M(\mathbb{C}^D) \to M(\mathbb{C}^D)$ (CP, non-TP)

$$T_s : \rho \mapsto \sum_i e^{sa_i} K_i \rho K_i^*$$

Central Limit Theorem for the measurement process

Theorem (Central Limit)

Let T be primitive. Then

1) *time averages converge to stationary means* $S_n(A) \to \mathbb{E}_{ss}(A)$

2) *fluctuations are normal*

$$\mathbb{F}_n(A) := \sqrt{n}(S_n(A) - \mathbb{E}_{ss}(A)) \xrightarrow[n\to\infty]{\mathcal{L}} N(0, V(A))$$

with variance

$$V(A) = \begin{cases} \mathbb{E}_{ss}(A^2) + 2\mathbb{E}_{ss}(A \otimes (\mathrm{Id} - T)^{-1}(B)), & B := \langle \chi | U^*(\mathbf{1} \otimes A) U | \chi \rangle \\ \frac{d^2 \log \lambda_s}{ds^2}, & \lambda_s = \text{dominant eigenvalue of } T_s \end{cases}$$

Remarks

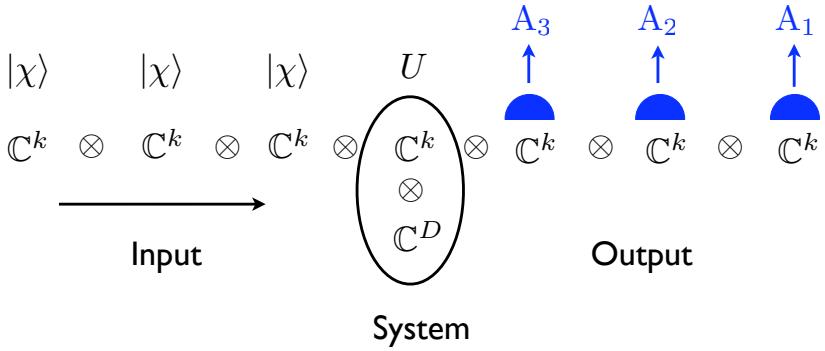
1) similar CLT holds for time averages of multiple-outcomes functions $f(A_1, \ldots, A_r)$[7]

2) similar CLT holds for the total counts and integrated homodyne current in continuous-time [8]

[7] M. van Horssen and M.G., J. Math. Phys. 2015

[8] C. Catana, L. Bouten, M.G., J. Phys. A, 2015

Classical Fisher information of time average



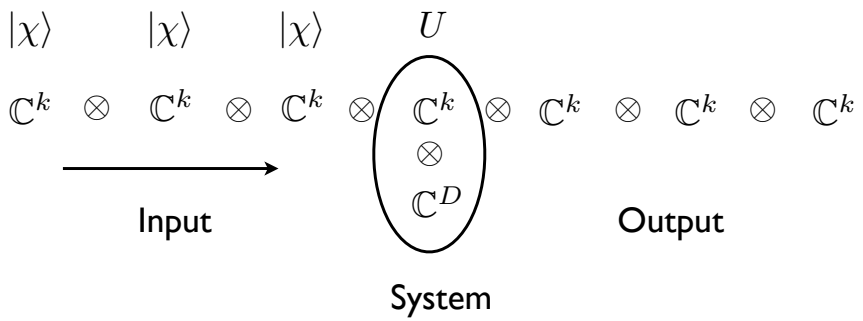
- Dynamics with isometry V_θ with unknown parameter $\theta = \theta_0 + u/\sqrt{n}$
- Time average $S_n = \frac{1}{n}\sum_{i=1}^n A_i$ captures deviations form mean $\mu_{\theta_0} = \mathbb{E}_{\theta_0}(A)$

$$\sqrt{n}(S_n - \mu_{\theta_0}) \xrightarrow{\mathcal{L}} N\left(\frac{d\mu}{d\theta}u, V(A)\right)$$

- Classical Fisher information = signal to noise ratio (in terms of dom. eigenv. $\lambda_{s,\theta}$ of $T_{s,\theta}$)

$$I^A(\theta_0) = \frac{\left(\frac{d\mu}{d\theta}\right)^2}{V(A)} = \frac{\left(\left.\frac{\partial^2 \lambda_{s,\theta}}{\partial s \partial \theta}\right|_{s=0,\theta=\theta_0}\right)^2}{\left.\frac{\partial^2 \lambda_{s,\theta}}{\partial s^2}\right|_{s=0}}$$

Quantum Markov dynamics



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- System-output state after n steps is of matrix product form[9]

$$|\Psi(n)\rangle = U(n)|\chi\rangle \otimes |\psi\rangle^{\otimes n} = \sum_{i_1,\ldots,i_n} K_{i_n} \ldots K_{i_1}|\chi\rangle \otimes |i_n\rangle \otimes \cdots \otimes |i_1\rangle$$

- If $U = U_\theta$, what is the quantum Fisher information of the output state ?

[9] M. Fannes, B. Nachtergale and R. Werner, 1992; D. Perez-Garcia, F. Verstraete, M. Wolf and I. Cirac, 2007

Linear scaling of quantum Fisher information (for primitive chains)

- One step generator $G_\theta := i\frac{dU_\theta}{d\theta} \cdot U_\theta^*$

- Total generator $G_\theta(n)$ as fluctuations operator

$$\begin{aligned} \frac{d}{d\theta}|\Psi_\theta(n)\rangle &= -iG_\theta(n)|\Psi_\theta(n)\rangle \\ G_\theta(n) &:= \sum_{i=1}^{n} U_\theta^{(n)} \dots U_\theta^{(i+1)} G_\theta^{(i)} U_\theta^{(i+1)*} \dots U_\theta^{(n)*} = \sqrt{n}\mathbb{F}_n(G_\theta) \end{aligned}$$

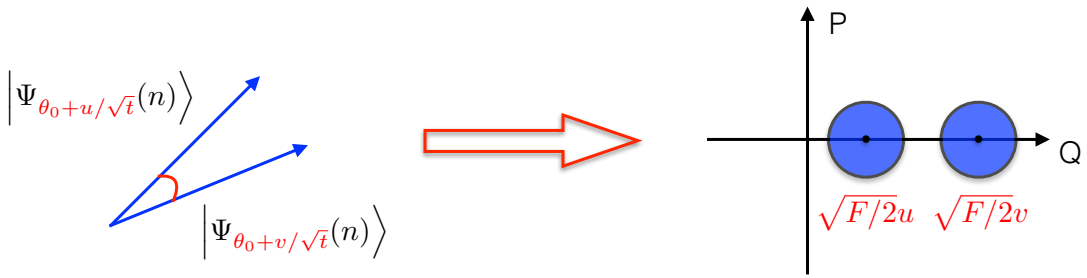
- QFI = Markov variance of generator $G_\theta(n)$ increases linearly with n and the constant is

$$\begin{aligned} F_\theta &:= \lim_{n\to\infty} \frac{1}{n} F_\theta^n = \lim_{n\to\infty} \frac{4}{n} \mathrm{Var}\left(G_\theta(n)\right) \\ &= 4\left\langle G_\theta^2 + 2\mathrm{Re}[G_\theta((\mathrm{Id}-T)^{-1}(K_\theta)\otimes \mathbf{1})]\right\rangle_{ss}, \quad K_\theta := \langle\chi|G_\theta|\chi\rangle \end{aligned}$$

- Remarks
  - linearity is related to the finite correlation time $\tau \propto \frac{1}{\mathrm{gap}}$ of the output
  - Constant in linear scaling may blow up as $(\mathrm{Id}-T)^{-1} \propto \frac{1}{\mathrm{gap}}$
  - Conjecture: all fluctuation operators satisfy Central Limit Theorem

Local asymptotic normality for the system + output state

- Primitive quantum Markov chain $V = V_\theta$ with θ unknown
- Localise parameter in a region of "uncertainty" size $\theta = \theta_0 + u/\sqrt{n}$



Theorem (LAN for quantum Markov chains)

The system + output quantum model $|\Psi_u^n\rangle := |\Psi_{\theta_0+u/\sqrt{n}}(n)\rangle$ *converges (weakly) to the quantum Gaussian shift model* $\sqrt{F/2}u\rangle$

$$\lim_{n\to\infty} \langle \Psi_u^n | \Psi_v^n \rangle = \left\langle \sqrt{F/2}u \middle| \sqrt{F/2}v \right\rangle = \exp(-F(u-v)^2/8)$$

- Remark
  - General LAN for the (mixed) output state $\rho_\theta(n) = \mathrm{Tr}_s(|\Psi_\theta(n)\rangle\langle\Psi_\theta(n)|)$[10]
  - LAN for all parameters with convergence to Gaussian shift on CCR algebra (–> Jukka)

[10] M.G. and J. Kiukas, Commun. Math. Phys. 2015

Quantum Fisher information as susceptibility

- Overlap can be reduced to dominant eigenvalue of a deformed transition operator

$$\langle \Psi_u^n | \Psi_v^n \rangle = \mathrm{Tr}\left(T^n_{\frac{u}{\sqrt{n}}, \frac{v}{\sqrt{n}}} (\rho_{\mathrm{in}}) \right) \approx \exp(n \log \lambda_{\frac{u}{\sqrt{n}}, \frac{v}{\sqrt{n}}})$$

- Expanding in $\frac{u}{\sqrt{n}}, \frac{v}{\sqrt{n}}$ and setting $\left.\frac{\partial \lambda_{a,b}}{\partial a}\right|_{a=b=0} = 0$

$$\langle \Psi_u^n | \Psi_v^n \rangle \longrightarrow \exp\left(\frac{1}{2} \left.\frac{\partial \log \lambda_{a,b}}{\partial a \partial b}\right|_{a=b=0} (u-v)^2 \right)$$

so that

$$F = -4 \left.\frac{\partial \log \lambda_{a,b}}{\partial a \partial b}\right|_{a=b=0}$$

- Similar methods have been used in [11]

[11] M. Cozzini, R. Ionicioiu, and P. Zanardi, Phys. Rev. B, 2007; S. Gammelmark and K. Mølmer, Phys. Rev. Lett., 2014

Equivalence classes with identical (stationary outputs)[14]

Definition

Two primitive chains with isometries V_1 and V_2 are called equivalent if for all n,

$$\rho^{out}_{V_1}(n) = \rho^{out}_{V_2}(n)$$

Theorem

Two primitive chains with isometries V_1 and V_2 are equivalent if and only if there exists a phase $e^{i\phi}$ and a unitary $W : \mathbb{C}^D \to \mathbb{C}^D$ such that

$$V_2 = e^{i\phi}(W \otimes \mathbf{1})V_1 W^*$$

or equivalently

$$K_i^{V_2} = e^{i\phi}(W \otimes \mathbf{1})K_i^{V_1} W^*, \qquad i = 1, \ldots, k.$$

Remarks

1) quantum extension of the "classical" result[12] on equivalence classes of ergodic hidden Markov chains

2) similar result holds in continuos-time: $L_i^{V_2} = W L_i^{V_1} W^*$ and $H^{V_2} = W H^{V_1} W^* + c\mathbf{1}$

3) similar result holds for (passive / active) linear systems[13]

[12] T. Petrie, Annals of Math. Statistics, 1969

[13] M.G. and N. Yamamoto, IEEE Trans. Control Theory (2016); M.Levitt M.G. arxiv:1608.01227

[14] M.G. and J. Kiukas, *Commun. Math. Phys.* 2015; M Fannes, B Nachtergaele, and R.F. Werner. *J. Funct. Anal.* 1994.

Idea of the proof

- Define the off-diagonal transition operator

$$T_{12} : \rho \mapsto \sum_{i=1}^{d} K_i^{V_1} \rho K_i^{V_2 *}$$

- Overlap of the two system-output states

$$\langle \Psi_{V_2 \psi}(n) | \Psi_{V_1 \psi}(n) \rangle \approx \lambda_{1,2}^n$$

- Two alternatives:

A) $|\lambda_{12}| = 1 \implies K_i^{V_2} = e^{i\phi} W K_i^{V_1} W^* \implies$ equivalent systems
B) $|\lambda_{12}| < 1 \implies$ overlap decays exponentially $\implies$ non-equivalent systems

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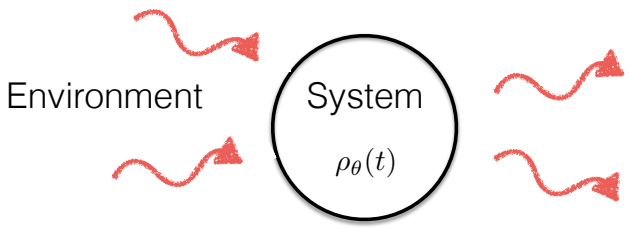
Markovian quantum open systems in continuous time



- Dissipative evolution of open system

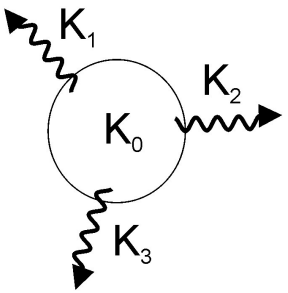
$$\frac{d}{dt}\rho(t) = \mathcal{L}\rho(t) = -i[H, \rho(t)] + \sum_i L^i \rho(t) L^{i*} - \frac{1}{2}\{\rho(t), L^{i*}L^i\}$$

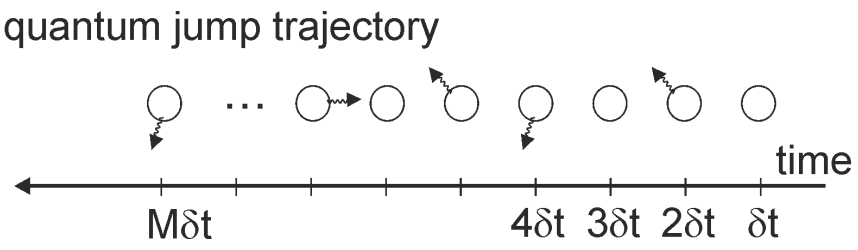
- Ergodicity: system converges to stationary state ρ_{ss} ($\mathcal{L}\rho_{ss} = 0$)

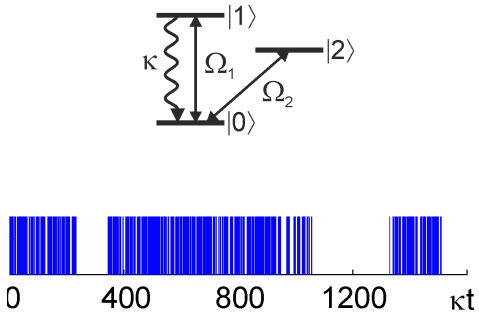
$$\rho(t) = e^{t\mathcal{L}}\rho_{in} \longrightarrow \rho_{ss}$$

- Estimate unknown "dynamical parameters" $\theta \mapsto D_\theta = (H_\theta, L^i_\theta)$ by measuring environment
  - system may not be accessible (e.g. in quantum control applications)
  - system would need to be initialised repeatedly
  - Information about dynamical parameters "leaks" continuously into the environment

Quantum trajectories = unravelling the master dynamics



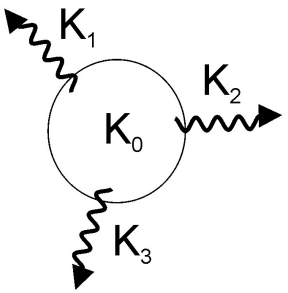


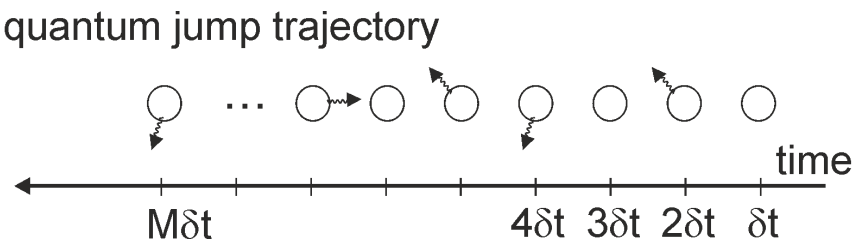


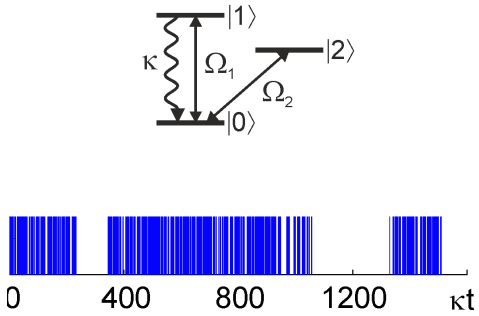
- Monitoring the environment produces jump trajectories with infinitesimal Kraus operators
  - "no emission": $K_0 = e^{-i\delta t H_\theta} \sqrt{1 - \delta t \sum_j L_\theta^{j*} L_\theta^j}$
  - "emission" in channel j: $K_j = e^{-i\delta t H_\theta} \sqrt{\delta t} L_\theta^j$

[15] M. Fannes, B. Nachtergale and R. Werner, 1992; D. Perez-Garcia, F. Verstraete, M. Wolf and I. Cirac, 2007
[16] K. R. Parthasarathy, *An introduction to quantum stochastic calculus*, Springer Birkhäuser, 1992

Quantum trajectories = unravelling the master dynamics







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- System-output state: coherent superposition of quantum trajectories, (continuous) MPS[15]

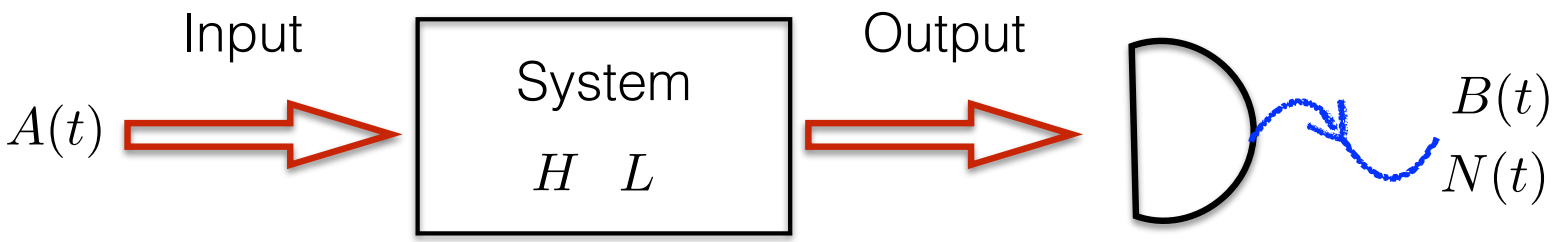
$$|\psi_\theta^{s+o}(t)\rangle = U_\theta(t)|\psi_{in}^{s+o}\rangle = \sum_{j_1,\ldots,j_n} K_{j_n} \ldots K_{j_1} |\chi\rangle \otimes |j_n \ldots j_1\rangle, \qquad n = t/\delta t$$

- Unitary dynamics: singular coupling with incoming input fields (Q Stoch Diff Eq[16])

$$dU_\theta(t) = \left(-iH_\theta dt + \sum_i L_\theta^i dA_i^*(t) - L_\theta^{i*} dA_i(t) - \frac{1}{2} L_\theta^{i*} L_\theta^i dt \right) U_\theta(t)$$

[15] M. Fannes, B. Nachtergale and R. Werner, 1992; D. Perez-Garcia, F. Verstraete, M. Wolf and I. Cirac, 2007
[16] K. R. Parthasarathy, *An introduction to quantum stochastic calculus*, Springer Birkhäuser, 1992

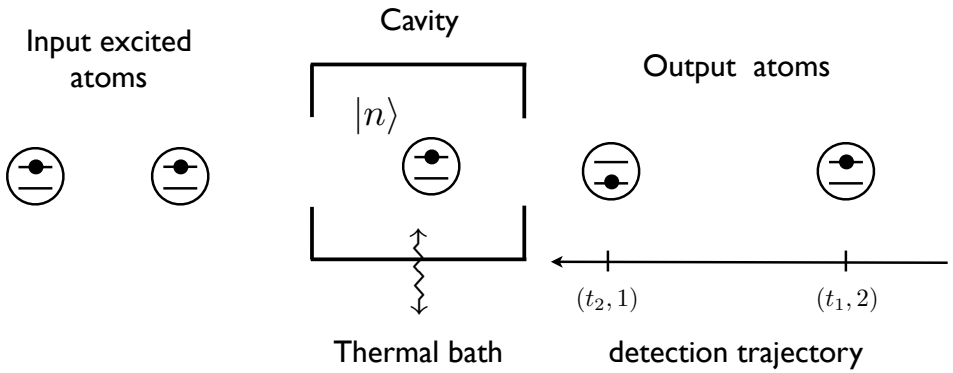
System identification and metrology with input-output open systems



- System identification: if $\theta \to (H_\theta, L_\theta)$, estimate θ by measuring the output[17]
  - which parameters can be identified ?
  - How does the output QFI scale with time t ?
  - How does this relate to dynamical properties, e.g. ergodicity, spectral gap...?
- Metrological power of output
  - is the Heisenberg limit achievable?
  - what is the short and long time behaviour ?
  - can the output be used as general purpose metrological probe state ?

[17] H. Mabuchi Quant. Semiclass. Optics (1996); J. Gambetta and H. M. Wiseman Phys. Rev. A (2001); S. Gammelmark and K. Molmer Phys. Rev. A (2013)...

Example: atom maser



- Atom maser with Jaynes-Cummings interaction

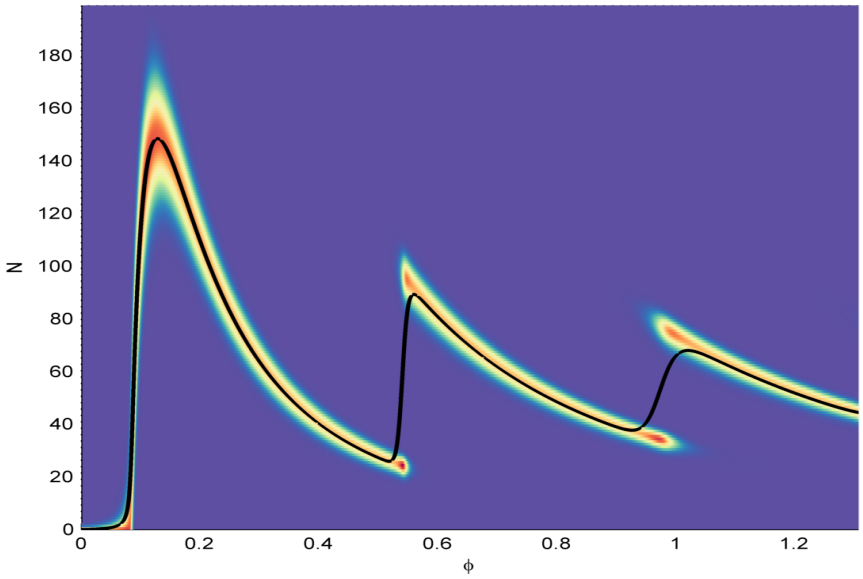
$$U : |1\rangle \otimes |k\rangle \mapsto \cos\left(\phi\sqrt{k+1}\right)|1\rangle \otimes |k\rangle + \sin\left(\phi\sqrt{k+1}\right)|0\rangle \otimes |k+1\rangle$$

- Coarse grained cavity dynamics for Poisson distributed input atoms with rate N_{ex}

$$\frac{d\rho}{dt} = \mathcal{L}(\rho) = \sum_{i=1}^{4}\left(L_i \rho L_i^* - \frac{1}{2}\{L_i^* L_i, \rho\}\right)$$

  - $L_1 : |k\rangle \mapsto \sqrt{N_{ex}}\sin(\phi\sqrt{k+1})\,|k+1\rangle$ (excitation absorbed from atom)
  - $L_2 : |k\rangle \mapsto \sqrt{N_{ex}}\cos(\phi\sqrt{k+1})\,|k\rangle$ (atom remains in excited state)
  - $L_3 : |k\rangle \mapsto \sqrt{k(\nu+1)}\,|k-1\rangle$ (photon emitted in the bath)
  - $L_4 : |k\rangle \mapsto \sqrt{(k+1)\nu}\,|k+1\rangle$ (photon absorbed from the bath)

Stationary state and phase transitions



Mean photon number and photon distribution in the stationary state as function of $\alpha = \sqrt{N_{ex}}\phi$

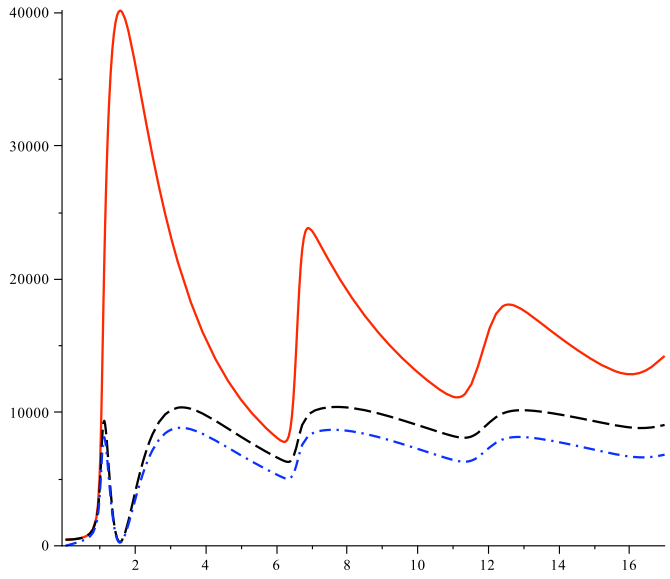
- unique stationary state

$$\rho_{ss}(n) = \rho_{ss}(0) \prod_{k=1}^{n} \left(\frac{\nu}{\nu+1} + \frac{N_{ex}}{\nu+1} \frac{\sin^2(\phi\sqrt{k})}{k} \right)$$

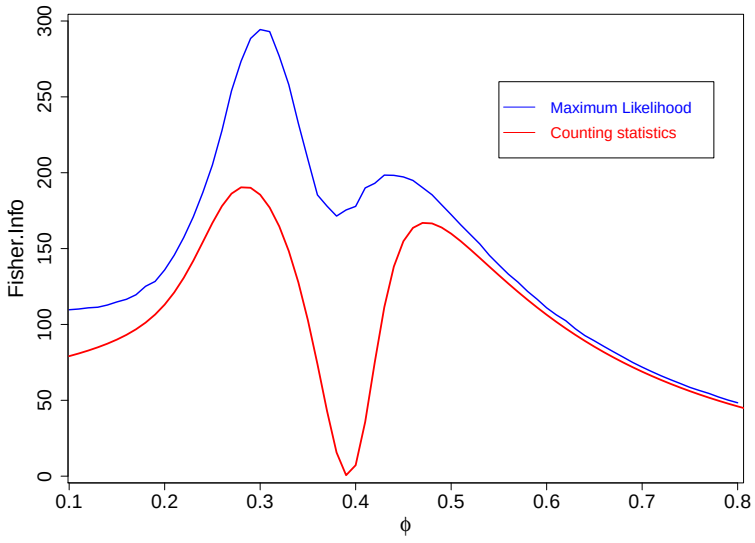
- jumps in mean photon number around $\alpha = 1, 2\pi, 4\pi$
- bistable stationary distribution around $\alpha = 2\pi, 4\pi$
- can be undestood via large deviations for the counting process[1]

[1] J. P. Garrahan and I. Lesanovsky, Phys. Rev. Lett. 2010

The many Fisher informations of the atom maser[18], [19]



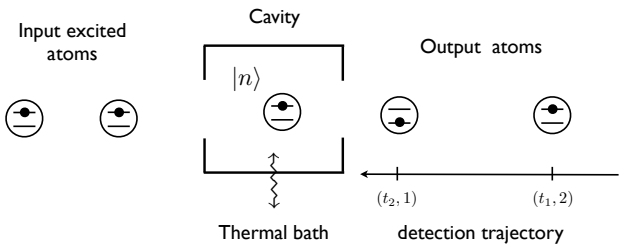
red: quantum Fisher info
black: observe cavity + bath
blue: observe cavity

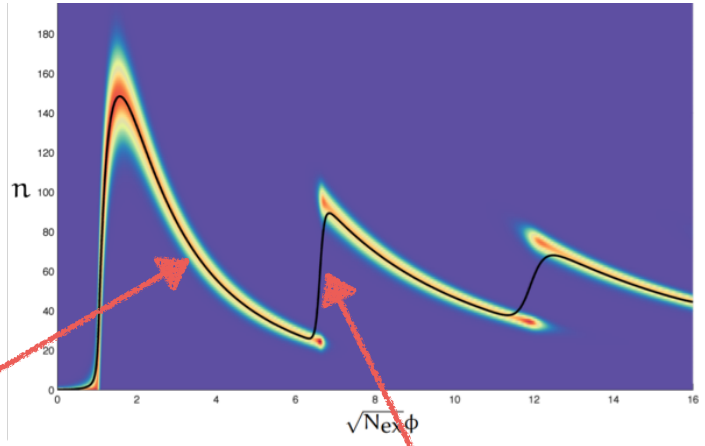


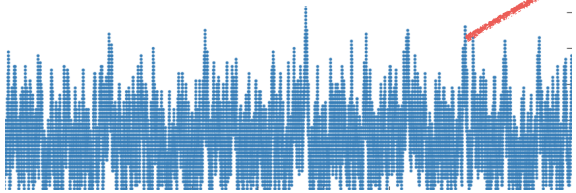
red: Fisher info total counts
blue: Fisher info counting process

[18] C. Catana, M van Horssen, M.G., *Phil. Trans. Royal Soc. A* 2012
[19] C.Catana, T. Kypraios and M.G. *J. Phys. A: Math. Theor.* 2014

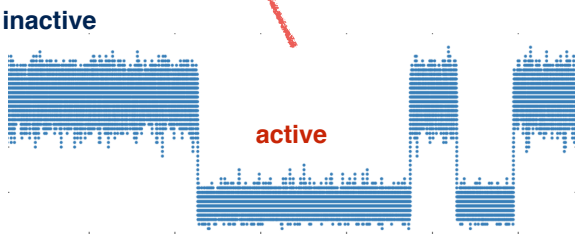
Thermodynamics of quantum trajectories [20] : the atom maser



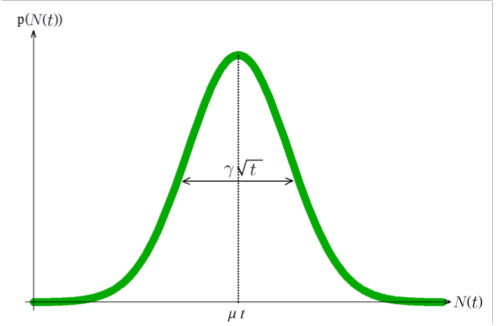




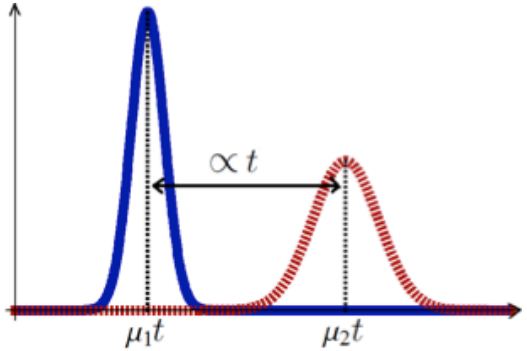
Cavity state (filter) away from DPT



Cavity state (filter) at DPT



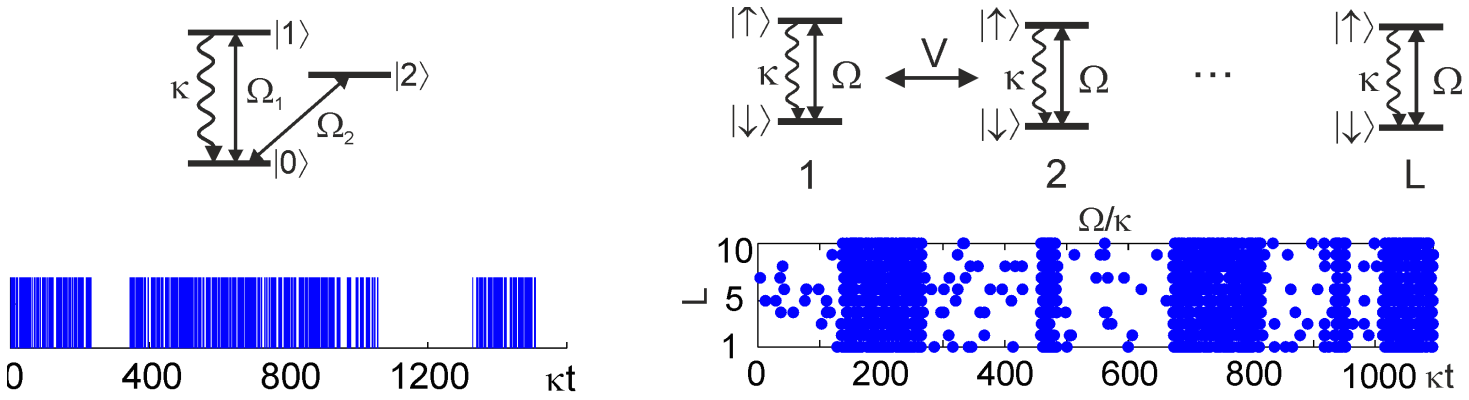
Counting distribution away from DPT



Counting distribution at DPT

[20] J. Garrahan, I. Lesanovsky, P.R.L. (2010)

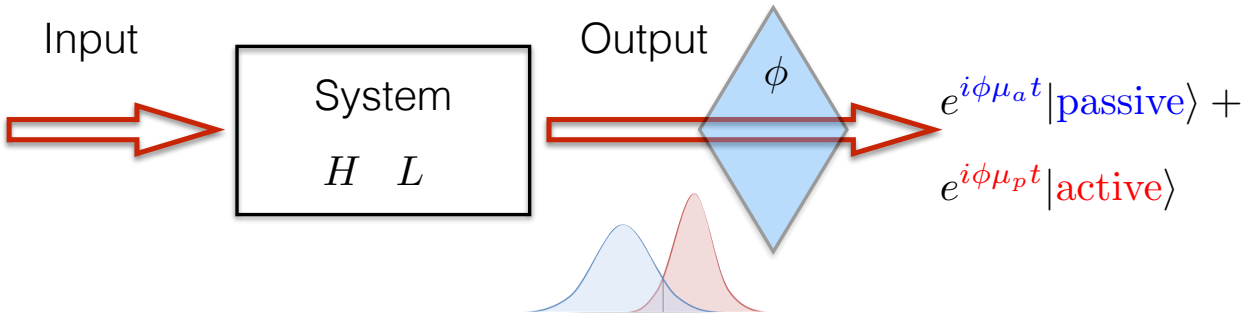
Counting statistics and dynamical phase transitions[21]



- If $\mathcal{L}$ is ergodic (spectral gap $\Delta\lambda := -\mathrm{Re}\lambda_2 > 0$) then
  - system converges to stationary state $\rho(t) = e^{t\mathcal{L}}(\rho_{\mathrm{in}}) \xrightarrow{t\to\infty} \rho_{ss}$
  - Counting operator $N(t)$ has normal fluctuations ($\Delta N(t) \propto \sqrt{t}$) around mean $t\mu$
- If $\mathcal{L}$ is near phase transition ($\Delta\lambda \approx 0$) then
  - metastability: slow convergence to stationarity, long correlation time $\tau = 1/\Delta\lambda$
  - intermittent trajectories, counting operator $N(t)$ has bimodal distribution up to times τ
- If $\mathcal{L}$ has degenerate stationary states then
  - infinite correlation times
  - counting operator $N(t)$ remains bimodal all times and variance increases as t^2

[21] J. Garrahan, I. Lesanovsky, P.R.L. (2010); I. Lesanovsky, M. van Horssen, M. G., J. P. Garrahan, P.R.L. (2013)

Phase estimation: Heisenberg limit at the DPT



- First order phase transition: system with two "stationary phases" ($\mathcal{H} = \mathcal{H}_i \oplus \mathcal{H}_a$) with different emission rates $\mu_i \neq \mu_a$
- Initial state: superposition $\sqrt{p_i}|\chi_i\rangle + \sqrt{p_a}|\chi_a\rangle$ with $|\chi_{a,i}\rangle \in \mathcal{H}_{i,a}$
- GHZ-type system-output state with generator $N(t)$

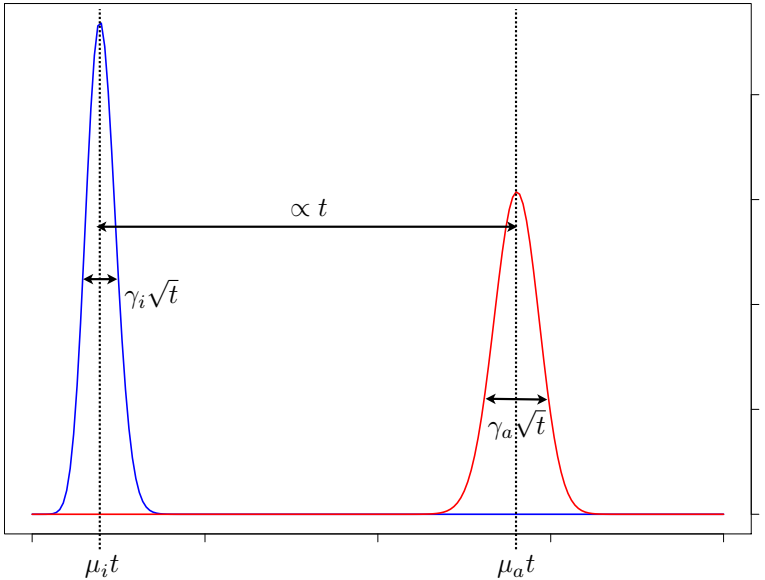
$$|\psi_\phi(t)\rangle = e^{i\phi N(t)}|\psi(t)\rangle \approx \sqrt{p_i}e^{i\phi\mu_i t}|\psi_i(t)\rangle + \sqrt{p_a}e^{i\phi\mu_a t}|\psi_a(t)\rangle$$

- Heisenberg limit wrt time:

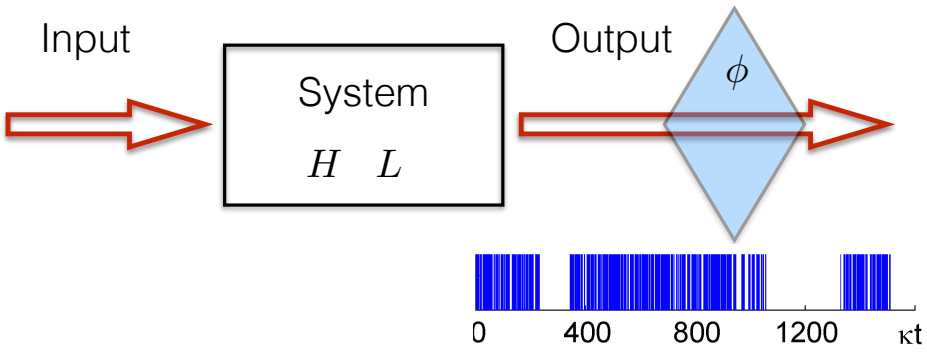
$$F(t) = 4\mathrm{Var}(N(t)) \approx t^2 p_i p_a (\mu_a - \mu_i)^2$$

- must measure sys+out to achieve QFI

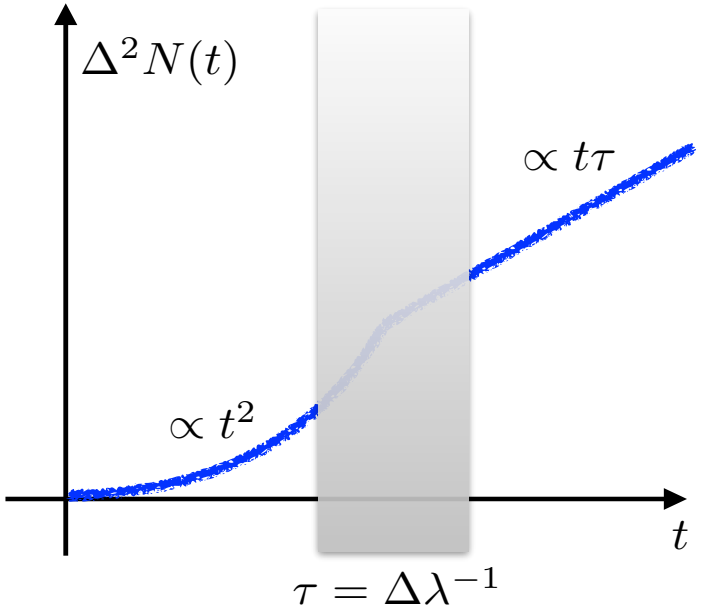
[K. Macieszczak, M.G. I. Lesanovsky, J. P. Garrahan Phys. Rev. A 2016]



Phase estimation: QFI time behaviour near phase transition



- System near first order DPT: metastability $\Longrightarrow$ counting trajectories exhibit intermittency
- Short time ($t \ll \tau$) distribution of generator $N(t)$ is bimodal $\Longrightarrow$ quadratic growth of QFI
- Long time ($t \gg \tau$) ergodicity and normal fluctuations win $\Longrightarrow$ linear growth of QFI



Conclusions and Outlook

- System identification: ergodic systems can be completely identified from output, up to unitary "change of coordinates"
- Asymptotic normality: output state is asymptotically Gaussian with quantum Fisher information equal to the "Markov variance of the generator"
- Systems at DPT exhibit Heisenberg scaling of estimation precisions
- Near phase transition: initial quadratic increase, then linear (possibly) with large constant

- Ongoing / future work
  - System identification and metrology for linear systems
  - Metastability theory for quantum open systems
  - General quantum Markov CLT
  - use of coherent feedback in system identification
  - design of better input states

References

M. Guta, J. Kiukas
Information geometry and local asymptotic normality for multi-parameter estimation of quantum Markov dynamics
ArXiv:1601.04355

K. Macieszczak, M. Guta, I. Lesanovsky, J. P. Garrahan
Dynamical phase transitions as a resource for quantum enhanced metrology
Phys. Rev. A 93, 022103 (2016)

C. Catana, L. Bouten and M. Guta
Fisher informations and local asymptotic normality for continuous-time quantum Markov processes
J. Phys. A, 48, 365301 (2015)

M. Guta, N. Yamamoto
System Identification for Passive Linear Quantum Systems
IEEE Trans. Automatic Control, (2016)

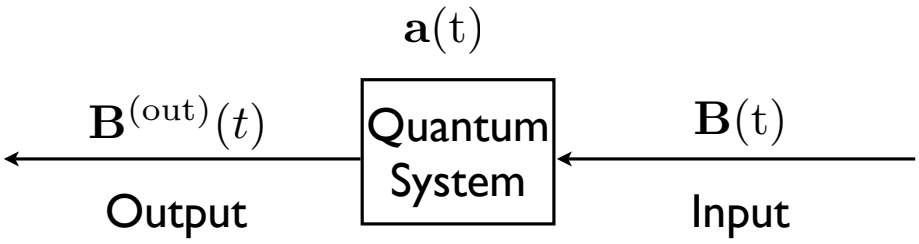
M. van Horssen and M. Guta:
Sanov and Central Limit Theorems for output statistics of quantum Markov chains
J. Math. Phys. 56, 022109 (2015)

M. Guta, J. Kiukas
Equivalence classes and local asymptotic normality in system identification for quantum Markov chains
Commun. Math. Phys. (2014)

I. Lesanovsky, M. van Horssen, M. Guta, J. P. Garrahan
Characterization of dynamical phase transitions in quantum jump trajectories beyond the properties of the stationary state
Phys. Rev. Lett. 110, 150401 (2013)

C. Catana, M. van Horssen, M. Guta:
Asymptotic inference in system identification for the atom maser
Phil. Trans. R. Soc. A 370, 5308-5323 (2012)

Linear quantum input-output systems



- System: n modes $\mathbf{a} = [a_1, \ldots, a_n]^T$ with Hamiltonian

$$\mathbf{H} = \mathbf{a}^\dagger \Omega_- \mathbf{a} + \frac{1}{2}\mathbf{a}^T \Omega_+^\dagger \mathbf{a} + \frac{1}{2}\mathbf{a}^\dagger \Omega_+ \mathbf{a}^*$$

- Input: m bosonic channels $\mathbf{B}(t) = [B_1(t), \ldots, B_m(t)]$
- Coupling: $\mathbf{L} = C_- \mathbf{a} + C_+ \mathbf{a}^*$
- Unitary quantum stochastic differential equations (QSDE)

$$dU(t) = \left(-i\mathbf{H}dt + \mathbf{L}d\mathbf{B}^\dagger(t) - d\mathbf{B}(t)\mathbf{L} - \frac{1}{2}\mathbf{L}^\dagger \mathbf{L}dt\right) U(t)$$

System identification problem:
estimate system matrices (Ω, C) from output measurements

Linear evolution

- Doubled-up notation: $\tilde{\mathbf{a}} = \begin{bmatrix} \mathbf{a} \\ \mathbf{a}^* \end{bmatrix} \ldots$

- Heisenberg evolution: $\tilde{\mathbf{a}}(t) := U(t)^* \tilde{\mathbf{a}} U(t)$ and $\tilde{\mathbf{B}}^{(\mathrm{out})}(t) := U(t)^* \tilde{\mathbf{B}}(t) U(t)$

$$\begin{aligned} d\tilde{\mathbf{a}}(t) &= \mathbf{A}\tilde{\mathbf{a}}(t)dt - \mathbf{C}^\sharp d\tilde{\mathbf{B}}(t) \\ d\tilde{\mathbf{B}}^{\mathrm{out}}(t) &= \mathbf{C}\tilde{\mathbf{a}}(t)dt + d\tilde{\mathbf{B}}(t) \end{aligned}$$

  - $\mathbf{C} = \Delta\left(C_-, C_+\right) \qquad \mathbf{A} = \Delta\left(A_-, A_+\right)$
  - $A_\mp = \frac{1}{2}\left(C_-^\dagger C_\mp - C_+^T C_\pm^*\right) - i\Omega_\mp$
  - $\Delta(X_-, X_+) = \begin{bmatrix} X_- & X_+ \\ X_+^* & X_-^* \end{bmatrix}$
  - $\mathbf{Z}^\sharp = J\mathbf{Z}^\dagger J$

- Input covariance / Ito rules (pure squeezed noise)

$$\begin{bmatrix} d\mathbf{B}d\mathbf{B}^\dagger & d\mathbf{B}d\mathbf{B}^T \\ d\mathbf{B}^*d\mathbf{B}^\dagger & d\mathbf{B}^*d\mathbf{B}^T \end{bmatrix} = \begin{bmatrix} \mathbf{N}+\mathbf{I} & \mathbf{M} \\ \mathbf{M}^\dagger & \mathbf{N} \end{bmatrix} dt = \mathbf{V}dt$$

Transfer function and power spectrum

- Laplace transform

$$\mathcal{L}[\mathbf{x}](s) := \int_0^\infty e^{-st}\mathbf{x}(t)dt, \qquad \mathrm{Re}s > 0$$

- Input-ouput relationship in Laplace domain

$$\mathcal{L}[\tilde{\mathbf{b}}^{(\mathrm{out})}](s) = \mathbf{F}(s) \cdot \mathcal{L}[\tilde{\mathbf{b}}](s)$$

where **F**(s) is the transfer function

$$\mathbf{F}(s) := \mathbf{I} - \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{C}^\sharp$$

- Power spectrum: output covariance in frequency domain

$$\Phi_V(-i\omega) = \mathbf{F}(-i\omega)\mathbf{V}\mathbf{F}(-i\omega)^\dagger$$

Identifiability classes in time dependent / stationary setting[22]

- minimal system: stable and cannot be reduced without changing the transfer function $\mathbf{F}(s)$
- globally minimal system: stable and cannot be reduced without changing the power spectrum $\Phi_V(s)$

Theorem

1. Let (Ω_1, C_1) and (Ω_2, C_2) be minimal systems with the same transfer function $\mathbf{F}(s)$. Then there exists a symplectic transformation T such that

$$J_n\Omega_2 = TJ_n\Omega_1T^\sharp, \;\; C_2 = C_1T^\sharp$$

2. A minimal system is globally minimal if and only if the stationary state is fully mixed.

3. Let (Ω_1, C_1) and (Ω_2, C_2) be globally minimal systems with the same power spectrum $\Phi_V(s)$. Then they have the same transfer function and are related by a symplectic transformation.

[22] M. Levitt, M.G., H. Nurdin arxiv: 1612.02681