Hypothesis testing for repeated quantum measurements and the emergence of the arrow of time

joint work with V. Jaksic, Y. Pautrat, C.-A. Pillet

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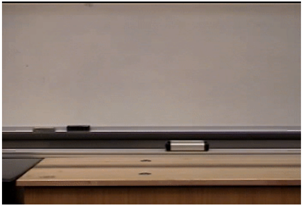
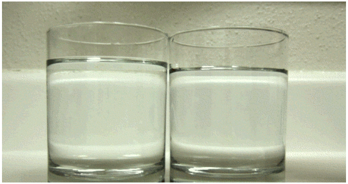
Toulouse, January 2017

Reversibility and irreversibility



Edited from Wake Forest University Physics Department video.



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Microscopic Reversibility and Macroscopic Irreversibility

Microscopic Reversibility:
Time reversal invariance: $\exists\theta$ an involution on the phase space such that,

$$O \circ f^{t-t_i}(x_{t_i}) = O \circ \theta \circ f^{t_f - t} \circ \theta(x_{t_f})$$

Example: $\theta(q) = q, \theta(p) = -p$.

Macroscopic Irreversibility:
Clausius (1850), thermodynamic:

$$Ep := \Delta S \geq 0,$$

Entropy always increases $\Rightarrow$ Thermodynamic time ordering.

Fluctuation relations

Developed in the nineties [Evans, Cohen, Gallavotti, Morris, Crooks . . .].

- $\mathbf{x} = (x_t)_t$: a path in phase space given by an Hamiltonian flow,
- $\sigma_t(x_0) = \sigma(x_t)$: entropy production rate random variable ($\mathbb{E}(\sigma_t) = \frac{1}{t} Ep_t$).

If the dynamical system is time reversal invariant, the transient fluctuation relation,

$$\frac{dP(\int_0^t \sigma_u du = s)}{dP(\int_0^t \sigma_u du = -s)} = e^{t\,s}$$

holds.
Under Chaotic Hypothesis, for any open set $O \subset \mathbb{R}$

$$\lim_{t\to\infty} \frac{1}{t} \log P\left(\tfrac{1}{t}\int_0^t \sigma_u du \in O\right) = -\inf_{s\in O} I(s),$$

With, $s \mapsto I(s)$ a good rate function such that:

$$I(s) \geq 0, \quad I(s) = 0 \iff s = Ep \quad \text{and} \quad I(s) = I(-s) - s.$$

Projection postulate and irreversibility

von Neumann (1932), 2 rules for quantum mechanics:

1. Projection postulate (PP): $\rho \to \rho' = \frac{P\rho P}{\mathrm{tr}[P\rho]}$ with proba $\mathrm{tr}[P\rho]$ *(Irreversible)*,
2. Unitary evolution: $\rho_t = e^{-iHt}\rho e^{iHt}$ *(Reversible)*.

Projection Postulate Irreversibility $\Rightarrow$ Quantum time ordering.

- Bohm (1951): *"This [quantum] irreversibility greatly resembles that which appears in thermodynamic processes"*.
- Landau-Lifschitz (1978): 2^{nd} law macroscopic expression of PP?

Criticism: Two state–vector formalism

Aharonov, Bergmann and Lebowitz (1964): *"This time asymmetry is actually related to the manner in which statistical ensembles are constructed"*

$$p(a \text{ then } j_1, j_2, \ldots, j_n; b) \neq p(b \text{ then } j_n, j_{n-1}, \ldots, j_1; a)$$

but,

$$p(j_1, j_2, \ldots, j_n; b|a) := |\langle b|j_n\rangle|^2 |\langle j_n|j_{n-1}\rangle|^2 \cdots |\langle j_2|j_1\rangle|^2 |\langle j_1|a\rangle|^2 = p(j_n, \ldots, j_2, j_1; a|b).$$

Conditioning on initial and final states, restores time reversal invariance (TRI) at the level of the measurement statistics.

Full Counting Statistics and Fluctuation Relations

Kurchan (2000):

$$H_i = \sum_{\epsilon} \epsilon |\epsilon\rangle\langle\epsilon|, \quad H_f = \sum_{\epsilon'} \epsilon' |\epsilon'\rangle\langle\epsilon'|$$

Work distribution:

$$P_t(W) = \sum_{W=\epsilon'_f - \epsilon_i} p(\epsilon_i \text{ then } \epsilon'_f) = \sum_{W=\epsilon'_f - \epsilon'_i} |\langle\epsilon'_f|U\epsilon_i\rangle|^2 \, \frac{e^{-\beta\epsilon_i}}{Z}.$$

$$\begin{array}{ccc} \rho \propto e^{-\beta H_i} \to |i\rangle\langle i| & U & U|i\rangle \to |f\rangle \\ 0 \;\vdash\!\!\!\longrightarrow & & \!\!\!\longrightarrow\!\!\!\dashv\; t \\ H_i \to \epsilon_i & W = \epsilon'_f - \epsilon_i & H_f \to \epsilon_f \end{array}$$

Time Reversal Invariance $\Rightarrow$ Crooks Fluctuation Relation: If,

$$\exists\Theta, \text{ s.t. } \Theta(\Theta(X)) = X, \quad \Theta(i\mathbb{1}) = -i\mathbb{1}, \quad \Theta(U) = U^*, \quad \Theta(\rho_i) = \rho_f$$

then,

$$\frac{dP_t(W = w)}{dP_t(W = -w)} = e^{\beta w}.$$

Entropic fluctuation relation

Entropy production defined as a two time measurement of the system state:

1. Mesure $-\ln \rho$: value s_i,
2. Evolve with $U := e^{-it(H+V)}$,
3. Measure $-\ln \rho$: value s_f.
4. **Entropy production:** $t\sigma := s_f - s_i$.

If time reversal invariance is verified:

$$\exists \Theta, \text{ s.t. } \Theta(i\mathbb{1}) = -i\mathbb{1}, \quad \Theta(\rho) = \rho, \quad \Theta(H) = H, \quad \Theta(V) = V.$$

Then the statistic of σ verifies:

$$S(\rho_t|\rho) = t\int_{\mathbb{R}} \sigma dP_t(\sigma) \quad \text{and} \quad \frac{dP_t(\sigma = s)}{dP_t(\sigma = -s)} = e^{t\,s}.$$

$\Rightarrow$ Positive entropy production is exponentially more likely.

Issue: Two time projective measurement of a non local quantity.
$\Rightarrow$ experimental propositions of measurement of the FCS using an auxiliary qbit interacting locally: Campisi, M. *et al.* New J. Phys. **15** (2013); Dorner, R. *et al.* PRL **110** (2013); Goold, J.*et al.* PRE **90** (2014); Mazzola, L. *et al.* PRL **110** (2013); Roncaglia, A.J. *et al.* PRL **113** (2014).

Entropy production of repeated measurements

Quantify: When can one choose the right movie order ? If it is possible how does the probability of error decays ?

$\implies$ Hypothesis Testing.

Why study repeated indirect measurements ?

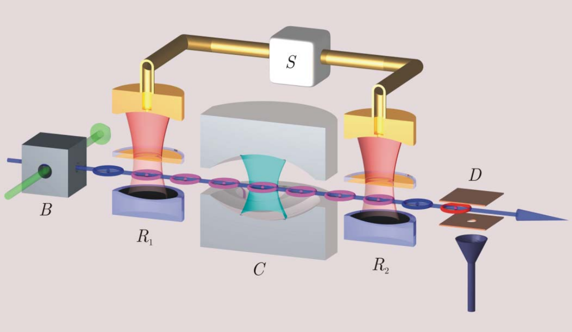
- Experimentally relevant (Cavity QED, Interferometry ...),
- "Every day experience",
- *Because we can have results.*

Outline

1. Repeated measurement model,
2. Instrument distinguishability and relative entropy
3. Rényi relative entropy regularity, (Fluctuation relations),
4. Hypothesis testing and error exponents.

A canonical experiment

S. Haroche group experiment:

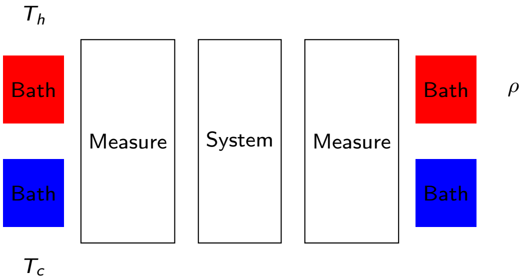


j	1101111111110011101101111
i	ddcbccabcdaadaabadddbadbc
j	0101001101010101101011111
i	dababbaacbccdadccdcbaaacc

j	0001000110110000001010110
i	ddcaddabbccdccbcdaabbccab
j	0001010100000100011101101
i	bcdaddaabbbbdbdcdccadaada

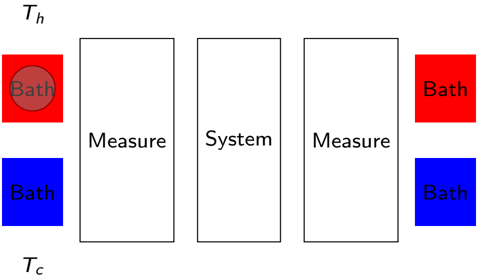
Image: LKB ENS

Repeated indirect measurements, a two heat baths example



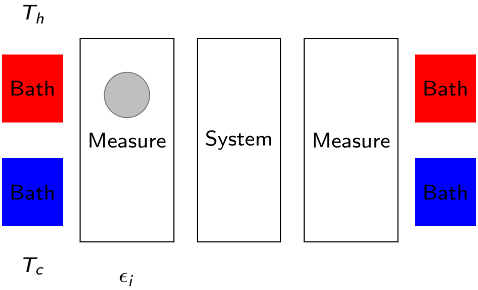
Hilbert space: $\mathcal{H} := \mathbb{C}^2_{sys.} \otimes \mathbb{C}^2_{hot} \otimes \mathbb{C}^2_{cold}$.

Repeated indirect measurements, a two heat baths example



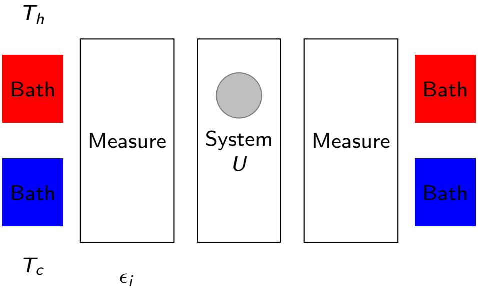
$\rho \otimes \beta_h$, prob. $\frac{1}{2}$

Repeated indirect measurements, a two heat baths example



$\rho \otimes |i\rangle\langle i|$, prob. $\langle i|\beta_h i\rangle$

Repeated indirect measurements, a two heat baths example



$$U(\rho \otimes |i\rangle\langle i|)U^*$$

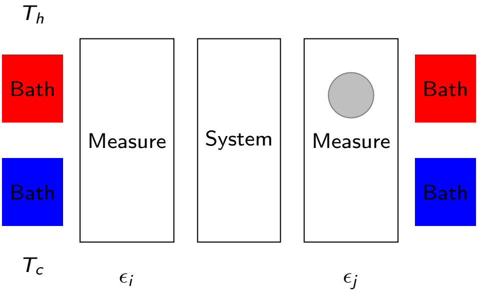
Dipolar, RWA:

$$U = \exp(i\tau H_{RWA})$$

$$H_{RWA} = \omega\sigma_z \otimes I + \omega I \otimes \sigma_z + \lambda\sigma_+ \otimes \sigma_- + h.c.$$

$$H_{full} = \omega\sigma_z \otimes I + \omega I \otimes \sigma_z + \lambda\sigma_x \otimes \sigma_x$$

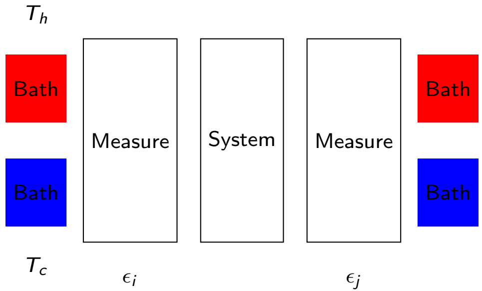
Repeated indirect measurements, a two heat baths example



$$\frac{U_{ji}\rho U_{ij}^*}{\mathrm{tr}[U_{ji}\rho U_{ij}^*]} \otimes |j\rangle\langle j|,$$

prob. $\mathrm{tr}[U_{ji}\rho U_{ij}^*] \times \langle i|\beta_h i\rangle$

Repeated indirect measurements, a two heat baths example



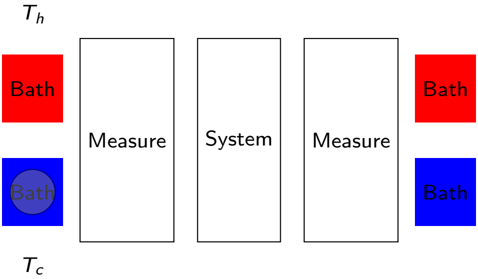
$$\rho(h;j,i) := \frac{V_{h;j,i}\,\rho\, V^*_{h;j,i}}{\mathrm{tr}[V^*_{h;j,i} V_{h;j,i}\,\rho]},$$

prob. $\mathrm{tr}[V^*_{h;j,i} V_{h;j,i}\,\rho]$.

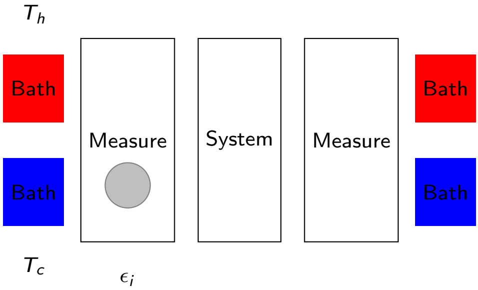
$$V_{h;j,i} = U_{ji}\sqrt{\langle i|\beta_h i\rangle},$$

$$\Delta E = \epsilon_j - \epsilon_i.$$

Repeated indirect measurements, a two heat baths example



Repeated indirect measurements, a two heat baths example



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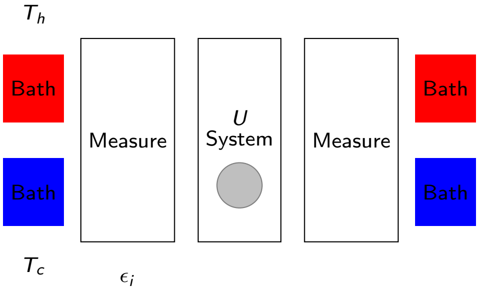
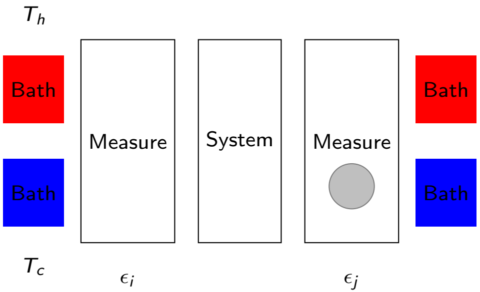
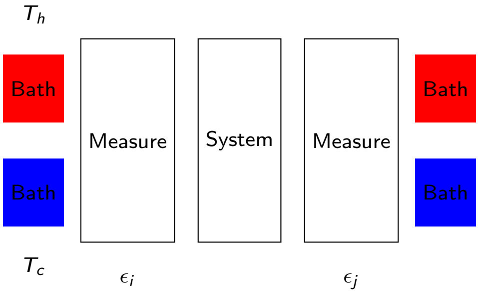


Repeated indirect measurements, a two heat baths example



Repeated indirect measurements, a two heat baths example



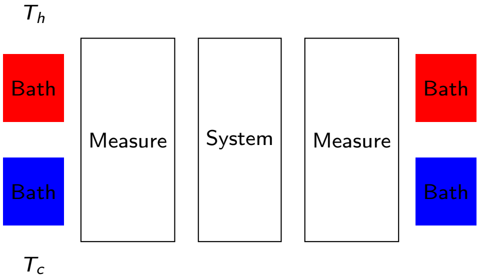
$$\rho(c;j,i) := \frac{V_{c;j,i}\,\rho\, V_{c;j,i}^*}{\mathrm{tr}[V_{c;j,i}^* V_{c;j,i}\,\rho]},$$

prob. $\mathrm{tr}[V_{c;j,i}^* V_{c;j,i}\,\rho]$.

$$V_{c;j,i} = U_{ji}\sqrt{\langle i|\beta_c i\rangle},$$

$$\Delta E = \epsilon_j - \epsilon_i.$$

Repeated indirect measurements, a two heat baths example



$$\rho(c;j,i) := \frac{V_{c;j,i}\rho V^*_{c;j,i}}{\mathrm{tr}[V^*_{c;j,i}V_{c;j,i}\rho]},$$

prob. $\mathrm{tr}[V^*_{c;j,i}V_{c;j,i}\rho]$.

Measurement result sequence: $((b_1;i_1,j_1),\dots,(b_t;i_t,j_t)) = (k_1,\dots,k_t)$ with probability

$$\mathbb{P}(k_1,\dots,k_t) = \mathrm{tr}[V_{k_t}\cdots V_{k_1}\rho V^*_{k_1}\cdots V_{k_t}].$$

Remark: Two time measurement process studied by Crooks[PRA '08] and Horowitz, Parrondo[NJP '13]

Quantum instruments

Definition (Instruments)

Let

$$\mathcal{J} := \{\Phi_k : M_d(\mathbb{C}) \to M_d(\mathbb{C})\}_{k=1,\dots,\ell}$$

be a familly of completely positive (CP) maps such that the CP map

$$\Phi := \sum_{k=1}^{\ell} \Phi_k$$

is unital (CPU). Then $\mathcal{J}$ is called an instrument.

Definition (Unraveling)

Let $\rho \in M_d^{+,1}(\mathbb{C})$ be a state on $\mathbb{C}^d$, then the probability measure $\mathbb{P}$ on $\Omega = \{1,\dots,\ell\}^{\mathbb{N}}$ defined by the marginals,

$$\mathbb{P}_t(k_1,\dots,k_t) = \mathrm{tr}(\rho \Phi_{k_1} \circ \cdots \circ \Phi_{k_t}(I_d))$$

is called an unraveling of the CPU map Φ.

Probability space

- Alphabet of the possible measurement outcomes: $\{1, \ldots, \ell\}$.
- Realization space: infinite sequences of letters, $\Omega = \{1, \ldots, \ell\}^{\mathbb{N}}$.
  Finite words:
  $$\Omega_t := \{1, \ldots, \ell\}^t, \quad \Omega_{\text{fin}} = \cup_{t \in \mathbb{N}} \Omega_t.$$
- σ–algebra: $\mathcal{F} := \sigma(\{\omega \in \Omega | \omega_s = k_s,\ 1 \leq s \leq n\})$,
- Measure on the words given by an instrument and a state: $(\mathcal{J}, \rho) \mapsto \mathbb{P}$,
  $$\mathbb{P}(k_1, \ldots, k_t) = \text{tr}(\rho \Phi_{k_1} \circ \cdots \circ \Phi_{k_t}(I_d)),$$
- Sequence shift: $f \circ \phi^t(\omega_0, \omega_1, \ldots) = f(\omega_t, \omega_{t+1}, \ldots)$.

Probability measure properties

Ergodic property:

- If ρ is the unique invariant state of Φ^*, then $(\Omega, \mathbb{P}, \phi)$ is ergodic.

$$\lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} \mathbb{E}(g\ f \circ \phi^t) = \mathbb{E}(g)\mathbb{E}(f).$$

From now on, we assume ρ is the unique invariant state of Φ^*.

Upper Bernoulli property:

- $\exists C > 0$ such that, for any finite word $k_1, \ldots, k_s, k_{s+1}, \ldots, k_t$,

$$\mathbb{P}(k_1, \ldots, k_s, k_{s+1}, \ldots, k_t) \leq C\mathbb{P}(k_1, \ldots, k_s)\mathbb{P}(k_{s+1}, \ldots, k_t).$$

Hypothesis testing

Given two possible instruments for an experiment $(\mathcal{J}, \widehat{\mathcal{J}})$ can we (and how efficiently can we) infer from the data $(\omega_s)_{s\in\mathbb{N}}$ what instrument is used?

H_0 The observed quantum measurements are described by $(\mathcal{J}, \rho)$.

H_1 The observed quantum measurements are described by $(\widehat{\mathcal{J}}, \widehat{\rho})$.

Arrow of time hypothesis testing

Let $\theta : \{1, \ldots, \ell\} \to \{1, \ldots, \ell\}$ be an involution (*i.e.* $\theta(\theta(k)) = k$).
A time reversal of the measurement results is then:

$$\Theta(k_1, \ldots, k_t) := (\theta(k_t), \ldots, \theta(k_1)).$$

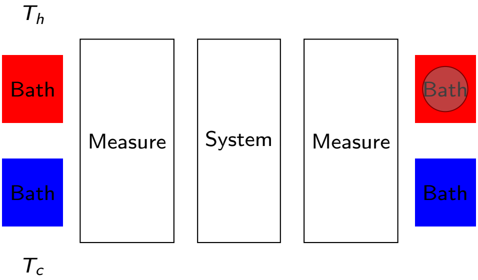
The time reversed probability measure over Ω is:

$$\widehat{\mathbb{P}}(k_1, \ldots, k_t) = \mathbb{P}(\theta(k_t), \ldots, \theta(k_1)).$$

$\widehat{\mathbb{P}}$ is the unraveling of $\widehat{\Phi}$ by the instrument $\widehat{\mathcal{J}} := \{\widehat{\Phi}_k\}_k$ with

$$\widehat{\Phi}_k(X) = \rho^{-\frac{1}{2}} \Phi^*_{\theta(k)}(\rho^{\frac{1}{2}} X \rho^{\frac{1}{2}}) \rho^{-\frac{1}{2}}.$$

Time reversal of the two baths example



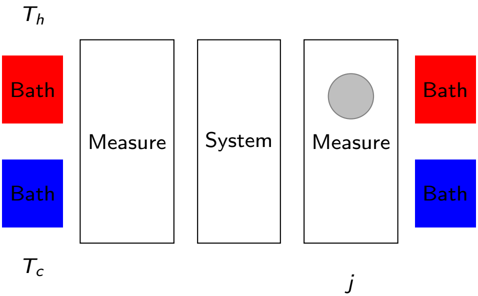
$\theta(b; i, j) = (b; j, i)$

$$((c, +\Delta E), (h, -\Delta E), \ldots, (h, 0))$$

reversed is

$$((c, -\Delta E), (h, +\Delta E), \ldots, (h, 0)).$$

Time reversal of the two baths example



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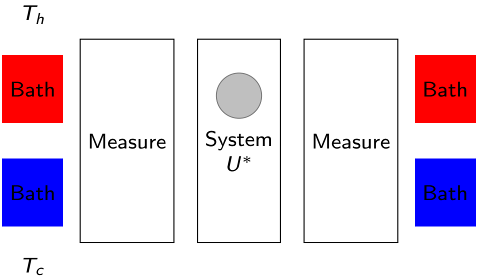

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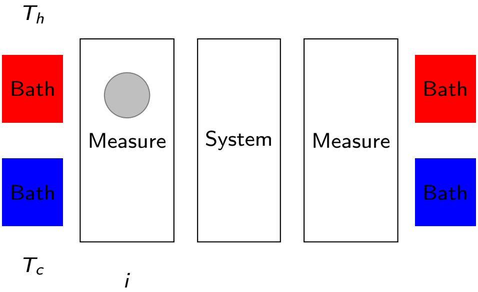

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$$((c, +\Delta E), (h, -\Delta E), \ldots, (h, 0))$$

reversed is

$$((c, -\Delta E), (h, +\Delta E), \ldots, (h, 0)).$$

Comparing $\mathbb{P}$ and $\widehat{\mathbb{P}}$

Assume non finite time distinguishability: $\mathbb{P}_t(A) > 0 \Leftrightarrow \widehat{\mathbb{P}}_t(A) > 0$ for all $t \in \mathbb{N}$.

We study the relative entropy,

- In mean:
$$S(\mathbb{P}_t|\widehat{\mathbb{P}}_t) := \sum_{\omega_t} \mathbb{P}_t(\omega_t) \log \frac{\mathbb{P}_t(\omega_t)}{\widehat{\mathbb{P}}_t(\omega_t)} \geq 0.$$
- As a random variable:
$$\sigma_t = \frac{1}{t} \log \frac{\mathbb{P}_t(\omega_t)}{\widehat{\mathbb{P}}_t(\omega_t)}.$$

Remark: For the time arrow, since the time reversal is an involution:

$$S(\mathbb{P}_t|\widehat{\mathbb{P}}_t) = \sum_{\omega_t \in \Omega_t} \mathbb{P}_t(\omega_t) \log[\mathbb{P}_t(\omega_t)/\widehat{\mathbb{P}}_t(\omega_t)] = S(\widehat{\mathbb{P}}_t|\mathbb{P}_t).$$

The relative entropy is then the entropy production and σ_t the entropy production rate.

Two sub additive convergence results

Lemma (Fekete)

Let $(a_t)_{t\geq 1}$ *be a sequence of real numbers such that for a* $c \in \mathbb{R}$ *and all* $s, t \in \mathbb{N}$*,*

$$a_{t+s} \leq a_t + a_s + c.$$

Then

$$\lim_{t\to\infty} \frac{1}{t} a_t = \inf_{t\geq 1} \frac{a_t + c}{t}.$$

Theorem (Kingman)

Let $X_t : \Omega \to \mathbb{R}$ *be a sequence of random variables such that* $\mathbb{E}(|X_t|) < \infty$*. Assume* $\exists C \in \mathbb{R}$ *such that for all* $t, s \in \mathbb{N}$*,*

$$X_{t+s}(\omega) \leq X_t(\omega) + X_s \circ \phi^t(\omega) + C$$

with $\mathbb{P}$ *probability* 1*. Then the limit*

$$x(\omega) := \lim_{t\to\infty} \frac{1}{t} X_t(\omega)$$

exists with probability 1 *and is* ϕ *invariant. Moreover*

$$\lim_{t\to\infty} \frac{1}{t} \mathbb{E}(X_t) = \mathbb{E}(x).$$

Relative entropy convergence

Theorem (B., Jaksic, Pautrat, Pillet '16)

Assume $\mathbb{P}$ and $\widehat{\mathbb{P}}$ are upper Bernoulli and ϕ-invariant. Then,

$$Ep := \lim_{t\to\infty} \frac{1}{t} S(\mathbb{P}_t|\widehat{\mathbb{P}}_t)$$

exists. Assume moreover that $\mathbb{P}$ and $\widehat{\mathbb{P}}$ are ergodic. Then,

$$\sigma := \lim_{t\to\infty} \sigma_t = \mathbb{E}(\sigma) = Ep. \quad \mathbb{P} - \textit{almost surely.}$$

Moreover,

$$Ep = 0 \Leftrightarrow \mathbb{P} = \widehat{\mathbb{P}} \quad \textit{and} \quad Ep > 0 \Leftrightarrow \mathbb{P}(\sigma > 0) = 1 \textit{ and } \widehat{\mathbb{P}}(\sigma > 0) = 0.$$

Relative entropy convergence

The asymptotic relative entropy random variable distinguishes between $\mathbb{P}$ and $\widehat{\mathbb{P}}$. Given an observed sequence of measurement $\omega \in \Omega$,

- Either $\sigma \geq 0$ and $\mathcal{J}$ is the instrument used (i.e. the movie is played forward),
- Or $\sigma \leq 0$ and $\widehat{\mathcal{J}}$ is the instrument used (i.e. the movie is played backward).

Remark

For time arrow hypothesis testing,

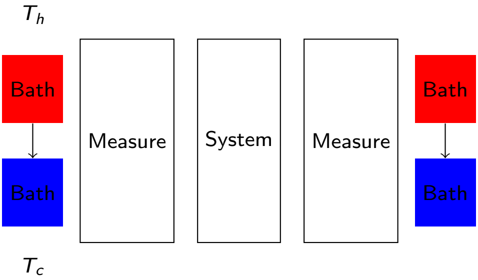
$$Ep = 0 \sim \textit{Detailed balance condition.}$$

With Detailed balance condition:

$$\Phi \equiv \widehat{\Phi}.$$

($\leftarrow$) *can be proved for a family of appropriate unraveling.*
($\rightarrow$) *partial from the theory of finitely correlated states [Fannes, Nachtergaele, Werner CMP '92].*

Entropy production



$$\sigma_t \simeq \frac{1}{t}(\frac{1}{T_h}\Delta Q_h + \frac{1}{T_c}\Delta Q_c).$$

Since $J_Q := \lim_{t\to\infty} \frac{1}{t}\Delta Q_c = \lim_{t\to\infty} -\frac{1}{t}\Delta Q_h$ with probability 1,

$$Ep = \frac{T_h - T_c}{T_c T_h} J_Q > 0 \quad \Rightarrow \quad \mathrm{sign}(J_Q) = \mathrm{sign}(T_h - T_c).$$

Since $J_Q = \Delta E \frac{1}{2}(\mathbb{P}(c; +\Delta E) - \mathbb{P}(c; -\Delta E))$,

$$Ep = 0 \Leftrightarrow T_c = T_h.$$

Remark: This is not true for full dipolar interaction where $Ep > 0$ even if $T_h = T_c$.

Beyond the law of large numbers: Rényi relative entropy.

Cumulant generating function of $-t\sigma_t$:

$$e_t(\alpha) := \log \sum_{\omega_t \in \Omega_t} \mathbb{P}_t(\omega_t)^{(1-\alpha)} \widehat{\mathbb{P}}_t(\omega_t)^{\alpha} = S_\alpha(\mathbb{P}_t | \widehat{\mathbb{P}}_t).$$

Remark: For the time arrow, since $\sum_{\omega_t} f(\omega_t) = \sum_{\hat{\omega}_t} f(\hat{\omega}_t)$,

$$e_t(\alpha) = e_t(1-\alpha).$$

Hence, the transient fluctuation relation holds: $\frac{\mathbb{P}_t(\sigma_t = s)}{\mathbb{P}_t(\sigma_t = -s)} = e^{t\,s}$.

Theorem (B.,Jaksic, Pautrat, Pillet '16)

Assume $\mathbb{P}$ and $\widehat{\mathbb{P}}$ are upper Bernoulli and ergodic. Then, $\forall \alpha \in [0,1]$,

$$e(\alpha) := \lim_{t\to\infty} \frac{1}{t} e_t(\alpha)$$

exists, is continuous, convex, satisfies $e(0) = e(1) = 0$ and

$$\partial_\alpha^+ e(\alpha)|_{\alpha=0} = -Ep.$$

For the time arrow hypothesis testing: $e(\alpha) = e(1-\alpha)$.

Rényi relative entropy as an entropic pressure

Rényi relative entropy can be obtained through a variational principle.

$$\frac{1}{t}e_t(\alpha) = \frac{1}{t}\max_{\mathbb{Q}_t}(\mathbb{E}_{\mathbb{Q}_t}(\log \mathbb{P}_t) - \alpha\mathbb{E}_{\mathbb{Q}_t}(\sigma_t) + S(\mathbb{Q}_t)).$$

Thermodynamic equivalent: Canonical Gibbs distribution maximises the free energy.

$$F_L \sim e_t(\alpha), \quad S_L \sim S(\mathbb{Q}_t) \quad \text{and} \quad \beta E_L \sim \alpha\mathbb{E}_{\mathbb{Q}_t}(\sigma_t) - \mathbb{E}_{\mathbb{Q}_t}(\log \mathbb{P}_t).$$

Since $-E_{t+s} \leq -E_t - E_s - C \Rightarrow$ sub additive thermodynamic formalism[1] $\Rightarrow$ regularity of $e(\alpha)$.

Let $\mathcal{P}_\phi$ be the set of ϕ invariant probability measures over Ω.
For all $\alpha \in [0,1]$ there exists $\mathbb{Q} \mapsto f_\alpha(\mathbb{Q})$ affine and upper semicontinuous such that:

$$e(\alpha) = \sup_{\mathbb{Q}\in\mathcal{P}_\phi} f_\alpha(\mathbb{Q})$$

Let $\mathcal{P}_{eq}(\alpha)$ be the set of probability measures for which the supremum is reached.
If $\mathcal{P}_{eq}(\alpha)$ is a singleton, then $\alpha \mapsto e(\alpha)$ is differentiable on $]0,1[$.

[1] [Barreira '10, Feng '09]

Differentiability of $e(\alpha)$

Assumption (C): (Weaker than lower Bernoulli) There exists τ and $C' > 0$ such that for all s, t, $\omega_t \in \Omega_t, \nu_s \in \Omega_s$, there exists $\xi_u \in \Omega_u$ with $u \leq \tau$ such that

$$\mathbb{P}(\omega_t, \xi_u, \nu_s)\widehat{\mathbb{P}}(\omega_t, \xi_u, \nu_s) \geq C' \mathbb{P}(\omega_t)\mathbb{P}(\nu_s)\widehat{\mathbb{P}}(\omega_t)\widehat{\mathbb{P}}(\nu_s).$$

Theorem (B., Jaksic, Pautrat, Pillet, '16)

If Assumption (C) holds, $\alpha \mapsto e(\alpha)$ is differentiable on $]0,1[$.

Assumption (D): (Quasi Bernoulli) There exists $C > 0$ such that for all s, t, $\omega_t \in \Omega_t, \nu_s \in \Omega_s$,

$$C^{-1}\mathbb{P}(\omega_t)\mathbb{P}(\nu_s) \leq \mathbb{P}(\omega_t, \nu_s) \leq C\mathbb{P}(\omega_t)\mathbb{P}(\nu_s).$$

Theorem (B., Jaksic, Pautrat, Pillet '16)

If both $\mathbb{P}$ and $\widehat{\mathbb{P}}$ verify Assumption (D), then, $\alpha \mapsto e(\alpha)$ exists and is differentiable on $\mathbb{R}$.

Sufficient condition for Assumption (C)

Proposition (B., Jaksic, Pautrat, Pillet '16)

Fix $(\mathcal{J}, \widehat{\mathcal{J}})$ two instruments on $M_d(\mathbb{C})$. Let $\mathcal{O} \subset M_{d^2}(\mathbb{C})$ be the smallest C^*−algebra containing the cluster points of the sequences

$$\left(\frac{\Phi_{\omega_t} \otimes \widehat{\Phi}_{\omega_t}(I_{d^2})}{\mathrm{tr}(\Phi_{\omega_t} \otimes \widehat{\Phi}_{\omega_t}(I_{d^2}))}\right)_{t\in\mathbb{N}} \quad \text{and} \quad \left(\frac{\Phi^*_{\omega_t} \otimes \widehat{\Phi}^*_{\omega_t}(I_{d^2})}{\mathrm{tr}(\Phi^*_{\omega_t} \otimes \widehat{\Phi}^*_{\omega_t}(I_{d^2}))}\right)_{t\in\mathbb{N}}$$

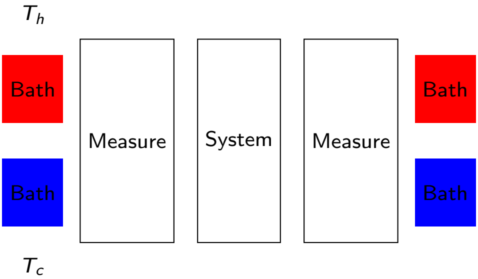
for any sequence of words $(\omega_t)_{t\in\mathbb{N}} \subset \Omega_{\mathrm{fin}}$ with $\Phi_\omega = \Phi_{\omega_1} \circ \cdots \circ \Phi_{\omega_{|\mathbf{a}|}}$.

If the CP map,

$$\Psi := \sum_{k=1}^{\ell} \Phi_k \otimes \widehat{\Phi}_k$$

is irreducible on $\mathcal{O}$, then Assumption (C) holds.

Rényi entropy and heat cumulant generating function.

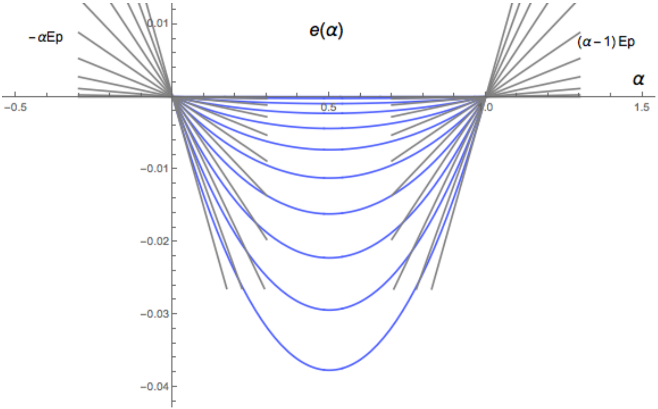


$e(\alpha)$ is the limit cumulant generating function of

$$-\sigma_t \simeq \frac{1}{t}\frac{T_c - T_h}{T_c T_h}\Delta Q_c.$$

It can be explicitly computed using spectral techniques on CP maps[van Horssen, Guta JMP '15].

$e(\alpha)$ for different $T_h - T_c$



$e(1/2)$ decreases when $T_h - T_c$ increases.

Fluctuation relation for time arrow testing

The entropy production random variable verifies a local large deviation principle.

$$I(s) := \sup_{\alpha \in \mathbb{R}} (\alpha s - \overline{e}(\alpha)).$$

From the symmetry $e(\alpha) = e(1-\alpha)$, this rate function is such that

$$I(-s) - I(s) = s \quad \text{and} \quad I(Ep) = 0.$$

Theorem

If Assumption (C) holds, for any $s \in]-Ep, Ep[$,

$$\lim_{\epsilon \downarrow 0} \lim_{t \to \infty} \frac{1}{t} \log \mathbb{P}_t(|\sigma_t - s| < \epsilon) = -I(s)$$

$$\lim_{\epsilon \downarrow 0} \lim_{t \to \infty} \frac{1}{t} \log \mathbb{P}_t(|\sigma_t + s| < \epsilon) = -I(-s) = -(I(s) + s)$$

If Assumption (D) holds, then both previous limit hold for any $s \in \mathbb{R}$.

Hypothesis testing

H_0 The observed quantum measurements are described by $(\mathcal{J}, \rho)$.

H_1 The observed quantum measurements are described by $(\widehat{\mathcal{J}}, \widehat{\rho})$.

For each time t let $\mathcal{T}_t$ be an event whose realization implies we decide "H_0 is true".

$$\text{Example: } \underline{\mathcal{T}}_t = \{\omega_t \in \Omega_t | \sigma_t > 0\}.$$

Then,

- $\mathbb{P}_t(\mathcal{T}_t^c)$ is the probability to reject H_0 when it is true (Type I error).
- $\widehat{\mathbb{P}}_t(\mathcal{T}_t)$ is the probability to accept H_0 when H_1 is true (Type II error).

Stein's error exponents

Stein error exponent for $\epsilon \in]0,1[$:

$$s_t(\epsilon) := \min_{\mathcal{T}_t}\{\widehat{\mathbb{P}}_t(\mathcal{T}_t) | \mathcal{T}_t \subset \Omega_t \text{ and } \mathbb{P}_t(\mathcal{T}_t^c) \leq \epsilon\}.$$

"$s_t(\epsilon)$ is the minimal error of type II while we control the error of type I."

Theorem (adapted from Jaksic, Ogata, Pillet, Seiringer '12)

Assume $\mathbb{P}$ and $\widehat{\mathbb{P}}$ are upper Bernoulli and ergodic. Then, for all $\epsilon \in]0,1[$,

$$\lim_{t\to\infty} \frac{1}{t} \log s_t(\epsilon) = -Ep.$$

The entropy production corresponds to the exponential decreasing rate of the error of type II given any control on the error of type I.

Hoeffding's error exponents

These exponents are similar to Stein's one, with a tighter control on the type I error.

$$\overline{h}(s) := \inf_{\mathcal{T}_t} \{ \limsup_{t\to\infty} \frac{1}{t} \log \widehat{\mathbb{P}}_t(\mathcal{T}_t) \mid \limsup_{t\to\infty} \frac{1}{t} \log \mathbb{P}_t(\mathcal{T}_t^c) < -s \}$$

$$\underline{h}(s) := \inf_{\mathcal{T}_t} \{ \liminf_{t\to\infty} \frac{1}{t} \log \widehat{\mathbb{P}}_t(\mathcal{T}_t) \mid \limsup_{t\to\infty} \frac{1}{t} \log \mathbb{P}_t(\mathcal{T}_t^c) < -s \}$$

$$h(s) := \inf_{\mathcal{T}_t} \{ \lim_{t\to\infty} \frac{1}{t} \log \widehat{\mathbb{P}}_t(\mathcal{T}_t) \mid \limsup_{t\to\infty} \frac{1}{t} \log \mathbb{P}_t(\mathcal{T}_t^c) < -s \}.$$

For $s \geq 0$, set

$$\Psi(s) = - \sup_{\alpha\in[0,1[} \frac{-s\alpha - e(\alpha)}{1-\alpha}.$$

Theorem (adapted from Jaksic, Ogata, Pillet, Seiringer '12)

Assume $\mathbb{P}$ and $\widehat{\mathbb{P}}$ are upper Bernoulli and ergodic, and that Assumption (C) holds. Then for $s \geq 0$,

$$\underline{h}(s) = \overline{h}(s) = h(s) = \Psi(s).$$

Chernoff's exponents

Assume a priori equiprobability for both hypothesis H_0 and H_1. Take the test,

$$\mathcal{T}_t = \{\omega_t \in \Omega_t | \sigma_t \geq 0\}.$$

Then, the total probability of error is:

$$c_t := \frac{1}{2}\mathbb{P}_t(\mathcal{T}_t^c) + \frac{1}{2}\widehat{\mathbb{P}}_t(\mathcal{T}_t) = \frac{1}{2}(1 - \|\mathbb{P}_t - \widehat{\mathbb{P}}_t\|_{TV}).$$

Chernoff exponents are:

$$\overline{c} := \limsup_{t\to\infty} \frac{1}{t} \log c_t \quad \text{and} \quad \underline{c} := \liminf_{t\to\infty} \frac{1}{t} \log c_t.$$

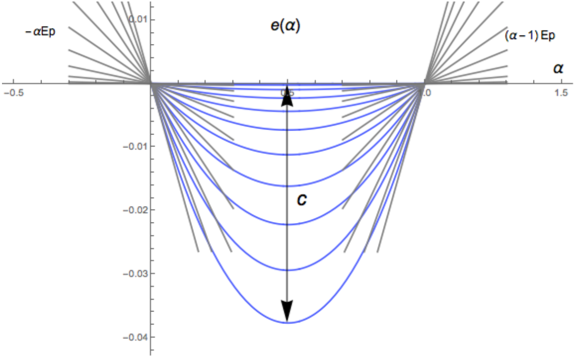
Theorem (adapted from Jaksic, Ogata, Pillet, Seiringer '12)

Assume $\mathbb{P}$ and $\widehat{\mathbb{P}}$ are upper Bernoulli and ergodic. Let $\alpha_* := \mathrm{argmin}_{\alpha\in[0,1]}\, e(\alpha)$. Then,

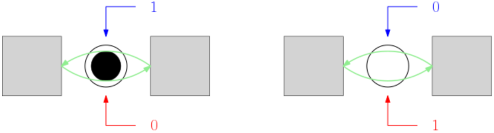
- $\overline{c} \leq e(\alpha_*)$ *and* $\underline{c} \geq e(\alpha_*) - \frac{1}{2}\partial^+ e(\alpha_*)$.
  Particularly $Ep > 0 \Rightarrow \overline{c} < 0$.
- *If Assumption (C) holds, $\overline{c} = \underline{c} = e(\alpha_*)$.*

Remark: *For time arrow hypothesis testing, $\alpha_* = \frac{1}{2}$.*

$e(\alpha)$ and error exponents



A non trivial example: The Keep–Switch instrument



Consider the two state Markov process with stochastic matrix $P := \begin{pmatrix} q_1 & p_1 \\ p_2 & q_2 \end{pmatrix}$. The two states are not accessible. One only knows if the particle following the Markov process stayed on its site (K) or switches site (S). The probability of sequences of keep and flip is given by the following instrument:
Let $\mathcal{J} := \{\Phi_K, \Phi_S\}$ with,

$$\Phi_K \left(\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \right) = \begin{pmatrix} q_1 x & 0 \\ 0 & q_2 y \end{pmatrix}, \quad \Phi_S \left(\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \right) = \begin{pmatrix} y p_1 & 0 \\ 0 & x p_2 \end{pmatrix}.$$

We do not know if our instrument gives a signal when the particle stays in its site or switches. The instrument $\widehat{\mathcal{J}}$ is $\widehat{\mathcal{J}} = \{\widehat{\Phi}_K = \Phi_S, \widehat{\Phi}_S = \Phi_K\}$.

We test the alternative $(\mathcal{J}, \widehat{\mathcal{J}})$.

A non trivial example: The Keep–Switch instrument

Proposition (B., Jaksic, Pautrat, Pillet '17)

1. *The Keep–Switch hypothesis testing verifies our assumptions and particularly Assumption (C).*
2. $\mathbb{P}$ *is not quasi Bernoulli (does not satisfy (D)).*
3. *The limit* $\alpha \mapsto e(\alpha)$ *of* $\alpha \mapsto e_t(\alpha)$ *exists and is differentiable on* $\mathbb{R}$*. It is real analytic on* $\mathbb{R} \setminus \{0, 1\}$ *but non twice differentiable in* 0 *and* 1*.*
4. *The CLT does not hold for the entropy production. Instead,*

$$\frac{\sigma_t - tEp}{\sqrt{t}} \xrightarrow[t\to\infty]{w} X + |Z|$$

with X *and* Z *two centered Gaussian random variables.*

Open questions

- Algebraic condition equivalent to Assumption (C),
- Φ irreducible and $\mathbb{P}_t \sim \widehat{\mathbb{P}}_t$ $\forall t$ such that $\alpha \mapsto e(\alpha)$ not differentiable on $]0,1[$,
- Irregularities outside $]0,1[$ and lower order Stein's exponents,
- Continuous time version,
- Entangled probes,
- Time reversal of the underlying Markov chain on the system.