Estimation of low-rank states by random measurements

Jonas KAHN
Cristina Butucea, Mădălin Guță

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Sparse

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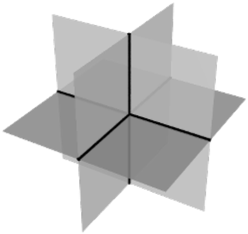

Compressed sensing

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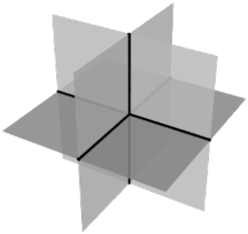
Sparse : belongs to a subspace of dimension s
We need (about) s measurements to find the parameter.



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Theorem (Cai and Zhang)

If the $X_i = f_i^* g_i$ are rank-one with f_i and g_i having independent Gaussian entries, we may rebuild A with about $r(p_1 + p_2)$ measurements.

Quantum setting

State

Density matrix $\rho \in M_d(\mathbb{C})$, non-negative, with trace 1.

Positive Operator-Valued Measure (POVM)

A measure with results in $(\mathcal{X}, \mathcal{A})$ is a set $\{M(A)\}_{A \in \mathcal{A}}$ of positive operators that are :

Positive $M(A) \geq 0$

Normalised $M(\mathcal{X}) = \mathbf{1}_{\mathcal{H}}$

Countably additive $M(\bigcup_{i \in \mathbb{N}} A_i) = \sum_{i \in \mathbb{N}} M(A_i)$ for all disjoint A_i

The $M(A)$ are called POVM elements.

Result of a measurement

The law of the result X of the measurement $\mathbf{M}$ applied to X the state ρ is

$$\mathbb{P}[X \in A] = \mathrm{Tr}[\rho M(A)].$$

Schatten norms

For symmetric matrices and $1 \leq p \leq \infty$, the Schatten p-norm $\|\cdot\|_p$ is given by

$$\|A\|_p^p = \sum_m |\lambda_m(A)|^p \,,$$

where the $\lambda_m(A)$ are the eigenvalues of A.

In particular, the Schatten 1-norm is the trace norm, the 2-norm is the Frobenius norm and the ∞-norm is the spectral norm.

Density matrices as classical mixture

For any set $\vec{\lambda} = \{\lambda_m\}_{1 \leq m \leq r}$ such that :

- $\lambda_m > 0$ for all i,
- $\sum \lambda_m = 1$,

we may consider the set of density matrices in $M_d(\mathbb{C})$ with those non-zero eigenvalues :

$$\mathcal{S}_{d,\vec{\lambda}} = \{\rho \in M_d(\mathbb{C}) : \lambda_m(\rho) = \lambda_m \delta_{m \leq r}\}.$$

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In particular r is the rank of ρ. Moreover,

$$\rho = \sum_m \lambda_m |\phi_m\rangle \langle\phi_m|$$

where the $|\phi_m\rangle$ are the eigenvectors of ρ.

Density matrices as classical mixture II

Measuring ρ with POVM M is equivalent to :

1. Choose m according to the probability distribution $\vec{\lambda}$.
2. Measure $|\phi_m\rangle \langle\phi_m|$ with POVM M.
3. Forget m.

Lower bound on estimation of low-rank quantum states

Proposition

Suppose we have n copies of a state $\rho \in \mathcal{S}_{d,\vec{\lambda}}$. For any estimator $\hat{\rho}$, the worst-case error has the lower bound :

$$\inf_{\hat{\rho}} \sup_{\rho \in \mathcal{S}_{d,\vec{\lambda}}} \mathbb{E}_\rho \left[\|\hat{\rho} - \rho\|_p \right] \geq C_p \sqrt{\frac{d}{n}} \left(\sum \lambda_m^{p/2} \right)^{1/p} .$$

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In particular, the corresponding lower bound on all rank r matrices is proportional to :

Trace norm $\sqrt{\frac{rd}{n}}$

Frobenius norm $\sqrt{\frac{d}{n}}$, not depending on $\vec{\lambda}$.

Spectral norm $\sqrt{\frac{d}{n}}$.

Totally random measurement

Definitions

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$$M(A) = \int_A |\psi\rangle \langle\psi| \, \mathrm{d}\mu(\psi),$$

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The result of the measurement is a rank-one state $|\psi\rangle \langle\psi|$.

Totally random measurement II

Suppose that the state ρ is a pure state $|e\rangle\langle e|$.
By symmetry, the result of the totally random measurement on ρ has expectation

$$\begin{aligned}\mathbb{E}_e\left[|\psi\rangle\langle\psi|\right] &= \alpha\,|e\rangle\langle e| + \beta\mathbf{1}\\ &= \alpha\rho + \beta\mathbf{1}\end{aligned}$$

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By linearity, the formula is still true if ρ is not rank-one.
Calculations yield

$$\mathbb{E}_\rho\left[|\psi\rangle\langle\psi|\right] = \frac{1}{d+1}\left(\rho + \mathbf{1}\right).$$

Totally random measurement III

- We have n copies of ρ.
- We measure each of them with the TRM.
- We get the results $|\psi_i\rangle\langle\psi_i|$ for $i \leq n$.

Recall

$$\mathbb{E}_\rho\left[|\psi\rangle\langle\psi|\right] = \frac{1}{d+1}\left(\rho + \mathbf{1}\right).$$

Natural estimator of ρ

$$\hat{\rho} = \qquad \frac{d+1}{n}\sum |\psi_i\rangle\langle\psi_i| - \mathbf{1}$$

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Totally random measurement estimator

Theorem

If $\rho \in \mathcal{S}_{d,r}$, then the risk of the totally random measurement procedure is bounded from above by :

$$\mathbb{E}_\rho\left[\|\hat{\rho} - \rho\|_p\right] \leq C_p r^{\frac{1}{p}} \sqrt{\frac{d}{n}}.$$

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In particular, it is minimax optimal for the Frobenius norm, up to a constant.

Proof

Write $X_i = |\psi_i\rangle\langle\psi_i| - \frac{1}{d+1}(\rho + \mathbf{1})$.

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Concentration inequality by Tropp

If X_i are independent centered Hermitian matrices with $\|X_i\|_\infty \leq W$ and $\left\|\mathbb{E}\left[\sum X_i^2\right]_\infty\right\| < V$, then for all $t > 0$:

$$\mathbb{P}\left[\left\|\sum X_i\right\|_\infty \geq t\right] \leq 2d\exp\left(-\frac{t^2/2}{V + tW/3}\right).$$

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Here $W = 1$ and $V = 2/d$. Hence

$$\mathbb{E}[\|R\|_\infty] \leq C\sqrt{\frac{d}{n}}.$$

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Since ρ has rank r, the error $\rho - \hat{\rho}$ has at most r positive eigenvalues. The error has trace zero. So that :

$$\|\hat{\rho} - \rho\|_1 \leq 2r\,\|\hat{\rho} - \rho\|_{\infty}.$$

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By interpolation,

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Stability to perturbations

$$\|\hat{\rho} - \rho\|_p \leq C_p\left(r + \sum_{i>r}\lambda_i(\rho)\sqrt{\frac{n}{d}}\right)^{1/p}\sqrt{\frac{d}{n}}.$$

Perspectives : multiple ions

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$\rho \in M_{2^b}(\mathbb{C})$ is the state of b qubits. Each qubit is measured individually using a Pauli observable. Easy.

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Do we get the same speeds ?

Perspectives : multiple ions II

Theorem (Xia, Koltchinskii ; Cai et al.)

$E_1, \ldots, E_{d^2}$ basis of the Hilbert space $M_d(\mathbb{C})$. Data is $\mathrm{Tr}\,(\rho E_i) + \xi_i$.
The squared minimax rate in Frobenius norm is $d^2/n = 4^b/n$.

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The Fisher information of the measurement on a relevant subspace would yield a $2^b/n$ bound.

Questions ?

Thank you !